1 October 25, 2016

Remember last time we had the following definition.

**Definition 1.1.** Let \( q : \mathcal{X} \to Y \) be a map from an algebraic stack \( \mathcal{X} \) to an algebraic space \( Y \). We say that \( q \) is a **good moduli space** (GMS) if

(i) \( q_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(Y) \) is exact

(ii) \( \mathcal{O}_Y \to q_* \mathcal{O}_X \) is an isomorphism.

We now state some properties of GMS maps.

**Proposition 1.2.** (1) Given a Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f'} & \mathcal{X} \\
\downarrow q' & & \downarrow q \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

(a) if \( f \) is fppf and \( q' \) is GMS, then \( q \) is GMS

(b) for \( f \) arbitrary, if \( q \) is GMS, then \( q' \) is GMS.

(2) If \( q : \mathcal{X} \to Y \) is GMS, then the canonical map \( F \to q_* q^*(F) \) is an isomorphism.

**Proof sketch.** (1a): The proof amounts to flat base change. We want to show that \( R^i q_*(E) = 0 \) for \( i > 0 \) and \( E \in \text{QCoh}(\mathcal{X}) \). It suffices by fpqc descent to show that \( f_* R^i q_*(E) = 0 \). There is a base change theorem which says that a flat map gives a base change equivalence. Thus \( f_* R^i q_*(E) \simeq R^i q'_*(f')^*(E) \).

(1b): Assume that \( f \) is affine. It is a fact that \( R^i f_*(E) = 0 \) for \( i > 0 \) when \( f \) is representable and affine. Because we desire to show that \( R^i(q'_*)_*(E) = 0 \), it suffices to show, since \( f_* : \text{QCoh}(Y') \to \text{QCoh}(Y) \) is faithful, that \( f_* (R^i(q'_*))(E) = 0 \). But we have the equivalences \( f_* (R^i(q'_*))_*(E) \simeq R^i (f \circ q'_*)_*(E) \simeq R^i q_*(f')^*(E) \).

(2): First reduce to the case where \( Y \) is affine (using flat base change). (Choose an fpqc map \( \bigcup \text{Spec}(A_i) \to Y \). The formation of \( q_* q^*(F) \) commutes with flat base change, as is the map \( F \to q_* q^* F \).

If \( Y \) is affine, we can find a presentation

\[
\mathcal{O}_Y^\oplus I \to \mathcal{O}_Y^\oplus J \to F \to 0
\]

where \( F \) is isomorphic to the cokernel of a map of free modules. This implies that

\[
\mathcal{O}_X^\oplus I \to \mathcal{O}_X^\oplus J \to q^* F \to 0
\]
is exact. Since $q_*$ is exact and $q_*\mathcal{O}_X = \mathcal{O}_Y$, we obtain exact

$$\mathcal{O}_Y^\oplus I \to \mathcal{O}_Y^\oplus J \to q_*q^* F \to 0.$$

As a result, we have the following.

**Corollary 1.3.**  
(i) If $I \subset \mathcal{O}_X$ is an ideal sheaf for a closed substack, then $q_*(\mathcal{O}_X/I) \simeq \mathcal{O}_Y/q_*I$.

(ii) $q_*(I_1) + q_*(I_2) = q_*(I_1 + I_2)$. (This follows from the fact that $q_*$ is an exact functor of abelian categories.)

(iii) If $J \subset \mathcal{O}_Y$ is an ideal sheaf and $I \subset \mathcal{O}_X$ is the preimage ideal sheaf, then the map $J \to q_*I$ is an isomorphism.

These statements about abelian categories have geometric consequences.

**Note 1.4.** For example, (ii) says that if $Z_1, Z_2 \hookrightarrow X$ are closed substacks, then $\text{im}(Z_1) \cap \text{im}(Z_2) = \text{im}(Z_1 \cap Z_2)$. This leads to an $S$-equivalence relation on geometric points, by saying that two geometric points map to the same points of $Y$ if and only if their closures intersect.

As a consequence of (iii), if $X$ is Noetherian, then $Y$ is Noetherian. Indeed, given an ascending chain of ideal sheaves on $Y$

$$J_1 \subset J_2 \subset \cdots \subset \mathcal{O}_Y,$$

we can take the preimages to obtain an ascending chain of ideal sheaves on $X$

$$I_1 \subset I_2 \subset \cdots \subset \mathcal{O}_X,$$

which stabilizes when $X$ is Noetherian. The pushforwards by $q_*$ must also stabilize eventually, which by (iii), implies that $J_n = q_*I_n$ stabilize as well.

**Corollary 1.5** (Hilbert 14). If $R$ is a finitely generated $G$-equivariant $k$-algebra and $G$ is linearly reductive, then $R^G$ is finitely generated.

**Proof.** Reduce to the case of a linear action $R = k[V]$ for some representation $V$ of $G$. (More precisely, there is a surjection $k[V] \to R$ which implies that $k[V]^G \to R^G$ is surjective because $G$ is linearly reductive, from which it follows that if $k[V]^G$ is finitely generated, then so is $R^G$.)

We have seen that $\text{Spec}(k[V])/G \to \text{Spec}(k[V]^G)$ is GMS. This implies that $k[V]^G$ is Noetherian.

The proof is complete from the fact that a graded ring $A = k \oplus \bigoplus_{n>0} A_n$ is finitely generated if and only if it is Noetherian, which is left as an exercise.

**Remark 1.6.** One idea for finding good moduli spaces is the following: Cover a stack $X$ by open substacks which have good moduli spaces themselves.
Example 1.7. The map Spec($R$)/$G \to$ Spec($R^G$) is always a GMS for any ring $R$ and any linearly reductive $G$. One does not even need $R$ to be finitely generated. One way to find an open substack is the following: for $f \in R^G$, then \{f \neq 0\} is $G$-equivariant. Another way is the following: if $\chi : G \to \mathbb{G}_m$ is a character and $f \in R$ is such that $g \circ f = \chi(g)f$ for each $g \in G$, then \{f \neq 0\} is also $G$-equivariant and affine; such $f$ is called \textbf{semi-invariant}. In fact, given $\chi : G \to \mathbb{G}_m$, can define Spec($R^\chi$) to be the set of points $x \in$ Spec($R$) such that there is a $\chi^n$ semi-invariant $f$ with $f(x) \neq 0$ for some $n > 0$. One can show that Spec($R^\chi$) is the union \[
bigcup_{f \in \chi$-semi-invariant} \text{Spec}(R[f^{-1}]).\] Moreover, Spec($R^\chi$)/$G$ has a GMS given by the map from Proj of the ring of $\chi^n$-semi-invariants to Spec($R^G$).