Main goal: cook up something close to a quotient space

\[ \mathcal{X} \to X \] universal for maps to alg. spaces

**Definition 4.1.** We say that \( \phi : \mathcal{X} \to Y \) is a **good moduli space** if the following properties are satisfied:

(i) \( \phi \) is cohomologically affine.
(ii) The natural map \( \mathcal{O}_Y \xrightarrow{\sim} \phi_* \mathcal{O}_\mathcal{X} \) is an isomorphism.

Simple looking definition, main properties

**Main Properties.** If \( \phi : \mathcal{X} \to Y \) is a good moduli space, then:

1. \( \phi \) is surjective and universally closed (in particular, \( Y \) has the quotient topology).
2. Two geometric points \( x_1 \) and \( x_2 \in \mathcal{X}(k) \) are identified in \( Y \) if and only if their closures \( \overline{\{x_1\}} \) and \( \overline{\{x_2\}} \) in \( \mathcal{X} \times_k k \) intersect.
3. If \( Y' \to Y \) is any morphism of algebraic spaces, then \( \phi_{Y'} : \mathcal{X} \times_Y Y' \to Y' \) is a good moduli space.
4. If \( \mathcal{X} \) is locally noetherian, then \( \phi \) is universal for maps to algebraic spaces.
5. If \( \mathcal{X} \) is finite type over an excellent scheme \( S \), then \( Y \) is finite type over \( S \).
6. If \( \mathcal{X} \) is locally noetherian, a vector bundle \( \mathcal{F} \) on \( \mathcal{X} \) is the pullback of a vector bundle on \( Y \) if and only if for every geometric point \( x : \text{Spec } k \to \mathcal{X} \) with closed image, the \( G_x \)-representation \( \mathcal{F} \otimes k \) is trivial.

(Alper '08)

The main example is \( G \) linearly reductive, \( R \) affine

\[ \text{Spec}(R)/\mathcal{C} \to \text{Spec}(RG) \]
\[
\text{Spec}(R)/G \rightarrow \text{Spec}(RG)
\]

In fact, this is a local model in great generality.

E.g., quotient stack whose good moduli space is a scheme locally of this form.

**Theorem 1.2.** Let \( \mathcal{X} \) be a quasi-separated algebraic stack, locally of finite type over an algebraically closed field \( k \), with affine stabilizers. Let \( x \in \mathcal{X}(k) \) be a point and \( H \subseteq G_x \) be a subgroup scheme of the stabilizer such that \( H \) is linearly reductive and \( G_x/H \) is smooth (resp. étale). Then there exists an affine scheme \( \text{Spec} \; A \) with an action of \( H \), a \( k \)-point \( w \in \text{Spec} \; A \) fixed by \( H \), and a smooth (resp. étale) morphism

\[
f : ([\text{Spec} \; A/H], w) \rightarrow (\mathcal{X}, x)
\]

such that \( BH \cong f^{-1}(BG_x) \); in particular, \( f \) induces the given inclusion \( H \to G_x \) on stabilizer group schemes at \( w \). In addition, if \( \mathcal{X} \) has affine diagonal, then the morphism \( f \) can be arranged to be affine.

**Theorem 2.9.** Let \( \mathcal{X} \) be a locally noetherian algebraic stack over \( k \). Suppose there exists a good moduli space \( X \) such that the moduli map \( \pi : \mathcal{X} \to X \) is of finite type with affine diagonal. If \( x \in \mathcal{X}(k) \) is a closed point, then there exists an affine scheme \( \text{Spec} \; A \) with an action of \( G_x \) and a cartesian diagram

\[
\begin{array}{ccc}
[\text{Spec} \; A/G_x] & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \pi \\
\text{Spec} \; A//G_x & \longrightarrow & X
\end{array}
\]

such that \( \text{Spec} \; A//G_x \to X \) is an étale neighborhood of \( \pi(x) \).
This leads to a strategy for constructing cover by affine quotient stacks!