Exercises VII

1. Construct subfields of $\mathbf{C}$ which are splitting fields of the following polynomial over $\mathbf{Q}$:

$$X^3 - 1, \ X^4 + 5X^2 + 2, \ X^6 - 8.$$

What are their degrees over $\mathbf{Q}$?

2. Construct the splitting fields over $\mathbf{F}_3$ for the following polynomials:

$$X^3 + 2X + 1, \ X^3 + X^2 + X + 2.$$

Are these fields isomorphic?

3. Which of the following extensions are normal?

$$\mathbf{Q}(X)/\mathbf{Q}, \ \mathbf{Q}(\sqrt{-5})/\mathbf{Q}, \ \mathbf{Q}(\alpha)/\mathbf{Q}, \ \mathbf{Q}(\alpha, \sqrt{5})/\mathbf{Q}(\alpha).$$

Here α is a real seventh root of 5.

4. Construct the normal closures of the following extensions:

$$\mathbf{Q}(\alpha)/\mathbf{Q}, \ \mathbf{Q}(\beta)/\mathbf{Q}, \ \mathbf{Q}(\sqrt{2}, \sqrt{3})/\mathbf{Q}, \ \mathbf{Q}(\gamma, \sqrt{2})/\mathbf{Q}, \ \mathbf{Q}(\delta)/\mathbf{Q}.$$

Here $\alpha^5 = 3$, $\beta^7 = 2$, $\gamma^3 = 2$ are real solutions and δ is a root of $X^3 - 3X^2 + 3$.

5. Find the Galois groups over $\mathbf{Q}$ of the extensions you constructed in exercise 4.