NAME:.................................. **18.702 Final exam.**

Preamble: This is a test exam. Except for the question on infinite Galois extensions it was the exam I gave 5 years ago in this class. Questions 4 (b), (c), (d) and (e) aren't relevant for this years course because we did not do Galois theory in characteristic $p > 0$. (Of course this doesn't mean I am not allowed to ask questions about fields in positive characteristic in this years exam, simply that I can't use the terms Galois, etc in char $p > 0$.

This is a closed book exam. Do not forget to put your name on this exam. Please write clearly. Explain your answers (except if you are asked not to explain). There are $\ntriangleleft$ questions on this exam.

1. Let (V, π) be a representation of the finite group G.
- **(a)** Define the character χ_π of π.
- **(b)** How can we read of the irreducibility of π from the character χ_π.
- **(c)** Show that if θ is the character of a 1 dimensional representation then $\theta \cdot \chi_\pi$ is the character of an irreducible representation if and only if π is irreducible.
- **(d)** Suppose $g \in G$ has order 2. What are the possible values of $\chi_\pi(g)$ if $\dim V = 3$?

2. Let G be a group with generators x, y, z, a, b and relations:

$$x^3 = y^3 = z^3 = a^3 = b^3 = 1,\ xy = yx,\ xz = zx,\ yz = zy,\ ab = ba,$$
$$axa^{-1} = xy,\ aya^{-1} = yz,\ aza^{-1} = z,\ bxb^{-1} = xz,\ byb^{-1} = y,\ bzb^{-1} = z.$$

This group has $3^5 = 243$ elements and the following conjugacy classes:

$$C_1 = \{1\},\ C_2 = \{z\},\ C_3 = \{z^2\},$$
$$C_4 = \{y, yz, yz^2\},\ C_5 = \{y^2, y^2z, y^2z^2\},$$
$$C_6 = \{x, xy, xz, xy^2, xyz, xz^2, xy^2z, xyz^2, xy^2z^2\},$$
$$C_7 = \{x^2, x^2y, x^2z, x^2y^2, x^2yz, x^2z^2, x^2y^2z, x^2yz^2, x^2y^2z^2\},$$

and for each pair (j, ℓ) with $j = 1, 2$ and $\ell \in \{0, 1, 2\}$ we have the classes

$$C^y_{(j,\ell)} = \{b^jy^\ell,\ b^jy^\ell z,\ b^jy^\ell z^2\},$$

and for each pair (j, k) with $j, k = 1, 2$ we have the conjugacy class

$$C^x_{(j,k)} = \begin{matrix}\{b^jx^k,\ b^jx^ky,\ b^jx^kz,\ b^jx^ky^2,\ b^jx^kyz, \\ b^jx^kz^2,\ b^jx^ky^2z,\ b^jx^kyz^2,\ b^jx^ky^2z^2\}\end{matrix},$$

and finally for each triple (i, j, k) with $i = 1, 2$ and $j, k \in \{0, 1, 2\}$ we have the conjugacy class

$$C_{(i,j,k)} = \begin{matrix}\{a^ib^jx^k,\ a^ib^jx^ky,\ a^ib^jx^kz,\ a^ib^jx^ky^2,\ a^ib^jx^kyz, \\ a^ib^jx^kz^2,\ a^ib^jx^ky^2z,\ a^ib^jx^kyz^2,\ a^ib^jx^ky^2z^2\}\end{matrix}.$$

Thus the total number of conjugacy classes is: $7 + 6 + 4 + 18 = 35$.

(a) What is the center of G?

(b) What is the maximal abelian quotient group of G? (Hints: The maximal abelian quotient of G is the quotient of G by the subgroup generated by commutators in G. The maximal abelian quotient has order 27.)

(c) How many distinct (irreducible) representations of dimension 1 does G have? (Hint: Relate one dimensional representations to the maximal abelian quotient of G.)

(d) Choose a non trivial representation of dimension 1 (i.e. one of the ones you counted above) and give its character values on the conjugacy classes described above.

(e) Show that besides the irreducible representations mentioned in (c) there have to be 2 irreducible representations of dimension 9 and 6 irreducible representations of dimension 3. (Hint: Numerology. You may use that the dimension of an irreducible representation divides the order of the group.)

(f) Let χ be one of the irreducible characters of dimension 9. Use the property of excercise 1(c) to show that $\chi(g)$ is zero for any $g \in G$ which maps to a nontrivial element of the maximal abelian quotient of G.

(g) For which conjugacy classes C of the list above does the result of (f) imply $\chi(g) = 0$ for all $g \in C$?

(h) Conclude that the other 9-dimensional irreducible representation cannot be obtained by tensoring the representation corresponding to χ with a 1-dimesional one.

(i) (Hard.) Show that the character χ has value $9\zeta_3$ or $9\zeta_3^2$ on z and determine the other character values from this. (Hint: z is central + use Schur's Lemma–also use the characterization of irreducible representations given in 1(b).)

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3. Let $F \subset K$ be a field extension.

(a) Define the characteristic of F.

(b) Define the degree of K over F.

(c) Assume that K is algebraic over F. State the condition that was used to define normality of K over F.

(d) Assume K is finite over F. Define the Galois property of the extension K/F.

4. Let $F \subset K$ be an extension of degree 2.

(a) Assume the characterisitic of F is not 2. Show that K can be obtained by adjoining a square root to F.

(b) Assume the characteristic of F is 2 and that K is NOT separable over F. Show that K is obtained by adjoining a square root to F.

(c) Assume the characteristic of F is 2 and that K is separable over F. Show K is Galois over F.

(d) Assumptions as in (c). Let $\sigma \in \mathrm{Gal}(K/F)$ be the nontrivial element. Show there exists an element $y \in K$ such that $\sigma(y) = y + 1$. (Hint: first pick $z \in K$, $z \notin F$ arbitrary and write $\sigma(z) = az + b$. Show $a = 1$ and then scale z to obtain y.)

(e) Assumptions and notations as in (d). Show that $y^2 + y \in F$. Conclude that every K/F as in (c) can be obtained by adjoining a root of $y^2 + y + a$ for some $a \in F$.

5. Let $f(X) = X^4 + 5X^2 + 3 \in \mathbb{Z}[X]$.

(a) What is the splitting field of this polynomial over $\mathbb{Q}$?

(b) What is the degree of this splitting field over $\mathbb{Q}$.

(c) What is the Galois group of this polynomial over $\mathbb{Q}$?

(d) Give the lattice of subgroups of this group. (This means: list all the subgroups and indicate the inclusion relations between them – usu. done in a kind of diagram.)

(e) Give the corresponding lattice of subfields of the splitting field.

(f) Indicate which of the subfields in (e) are Galois over $\mathbb{Q}$, and what are their Galois groups over $\mathbb{Q}$.

Recall: (Infinite) Galois extensions are defined as normal algebraic extensions in char 0. The Galois group is simply the automorphism group of the extension. In the exercise below you will (I think) have to use a theorem on (infinite) Galois extensions that I proved in the lectures...

6. In this exercise all fields are subfields of $\mathbb{C}$. Let $L = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots)$. In other words, L is the field you get from $\mathbb{Q}$ by adjoining all square roots (in $\mathbb{C}$) of rational numbers.

(a) Let f be an irreducible polynomial over $\mathbb{Q}$ of degree $n \geq 5$ with Galois group S_n. Show that the Galois group of f over the field L is A_n.

(b) Suppose that f_n is a sequence of such polynomials, one for each $n \geq 5$. Show that the field extension $L \subset M$ you get by adjoining all the complex roots of all f_n to L is an infinite Galois extension. (I mean it is definitively not finite.)

(c) Show that the center of the group $\mathrm{Gal}(M/L)$ is trivial.

(d) Guess the structure of the Galois group $\mathrm{Gal}(M/L)$.