A Postgraduate course taught by
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Columbia University, Fall 2008
Notes taken by Qi You
"Are there any undergraduate students?" Asked de Jong.
One student in the back raised his hand.
"Generally I don't encourage undergrads to take algebraic geometry.
You know, it could be, could be, you know..."
He paused for a second.
"It's just like we should never talk about sex in front of kids!"
§1. The Spectrum of a Ring

Def. Let \( R \) be a ring.

1. The spectrum of \( R \) is the set of prime ideals of \( R \), denoted \( \text{Spec} \, R \).
2. Given a subset \( T \subseteq R \), we let \( V(T) \) in \( \text{Spec} \, R \) be the set of prime ideals containing \( T \): \( V(T) = \{ \mathfrak{p} \in \text{Spec} \, R \mid \mathfrak{p} \supseteq T \} \).
3. Given an element \( f \in R \), we let \( D(f) \subseteq \text{Spec} \, R \) be the set of primes \( \mathfrak{p} \) not containing \( f \).

Lemma:

1. \( \text{Spec} \, R \) is empty iff \( R = \{0\} \).
2. Every nonzero ring has a maximal ideal.
3. Every nonzero ring has a minimal prime ideal.
4. Given \( I \subseteq R \) and a prime ideal \( \mathfrak{p} \supseteq I \), then there exists a prime ideal \( \mathfrak{q} \) with \( \mathfrak{q} \subseteq \mathfrak{p} \) and \( \mathfrak{q} \) minimal over \( I \).
5. If \( T \subseteq R \), then \( V(I) = V(T) \).
6. Given \( I \subseteq R \), \( \sqrt{I} = \{ f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N} \} = \bigcap \mathfrak{p} \supseteq \mathfrak{p} \).
7. If \( I \) is an ideal, then \( V(I) = V(\sqrt{I}) \).
8. If \( I \) is an ideal, and \( V(I) = \emptyset \), then \( I = R \) and \( V(0) = \text{Spec} \, R \).
9. \( I, J \) ideals, then \( V(I) \cup V(J) = V(I \cap J) = V(IJ) \).
10. If \( \{ I_i \}_{i \in A} \) is a collection of ideals, \( \bigcap_{i \in A} V(I_i) = V(\bigcap_{i \in A} I_i) \).
11. If \( f \in R \), \( \text{Spec} \, R = V(f) \cup D(f) \).
12. If \( f \in R \), then \( D(f) = \emptyset \) if \( f \) is nilpotent.
13. If \( f, f' \in R \), \( f = uf' \) with \( u \) a unit \( \Rightarrow D(f) = D(f') \).
14. If \( I \) is an ideal, \( \mathfrak{p} \) is a prime, \( \mathfrak{p} \in V(I) \), then \( \exists f \in I \text{ s.t. } \mathfrak{p} \in D(f) \) and \( D(f) \cap V(I) = \emptyset \).
15. \( f, g \in R \Rightarrow D(f) \cap D(g) = D(fg) \).

Def. Let \( R \) be a ring. The topology on \( \text{Spec} \, R \) whose closed sets are \( V(T) \) is called the Zariski topology (18), (19), (20) of lemma .... The open sets \( D(f) \) are called standard open. They form a basis of the Zariski topology.
Proof of lemma:

(1) $^\Rightarrow$ easy.

(2) $^\Rightarrow$ $R 
eq 0$ in $R$, by (3), Spec$R$ is not empty.

(3) $R$ non-zero $\Rightarrow 1 \neq 0$ in $R$. Let $\mathcal{I} = \{I \in \text{ideals of } R | \text{Spec} R \text{ is not considered as an ideal}\}$.

As a partially ordered set where $\leq$ is defined by $I \subseteq I'$, $\mathcal{I}$ is not empty since $(0) \in \mathcal{I}$. Thus if we can show that if any totally ordered subset of $\mathcal{I}$ has an upper bound, then $\mathcal{I}$ will have a maximal element by Zorn's lemma, which is nothing but a maximal ideal. Indeed, let $(I_a)_{a \in A}$ be a totally ordered subset of $\mathcal{I}$.

Claim: $\bigcup_{a \in A} I_a \in \mathcal{I}$.

Clearly $1 \in \bigcup_{a \in A} I_a$. Also if $x, y \in \bigcup_{a \in A} I_a \Rightarrow x \in I_a, y \in I_a$ for $a, a' \in A$. Since $A$ is totally ordered, W.L.O.G. $a \leq a' \Rightarrow I_a \subseteq I_{a'} \Rightarrow x, y \in I_a \Rightarrow x + y \in I_a \subseteq \bigcup_{a \in A} I_a$. Moreover, $ax \in I_a \subseteq \bigcup_{a \in A} I_a$. Thus $\bigcup_{a \in A} I_a$ is an ideal and is an upper bound for $\mathcal{I}$.

(4) Similar proof as in (3), with the partial relation changed into $I \subseteq I' \Rightarrow I \subseteq I'$ and $\mathcal{I}$ is non-empty since we have maximal ideals by (3).

"Let me tell you something about sets"

Remarked de Jong. Everybody concentrated.

"It's very big~~~"

(5). Apply (3) to the ring $A/I$

(6). Any $\mathfrak{p} \supseteq \mathfrak{I}$ certainly contains $\mathfrak{T}$

(6). Let $\mathfrak{p}$ be in $\text{V}(\mathfrak{I})$. If $x \in \mathfrak{I} \Rightarrow \mathfrak{x} \in \mathfrak{I} \Rightarrow \mathfrak{x} \in \mathfrak{p} \Rightarrow \mathfrak{x} \in \mathfrak{p} \Rightarrow \mathfrak{I} \subseteq \bigcap_{x \in \mathfrak{I}} \mathfrak{p}$

Conversely, $x \in \mathfrak{I} \Rightarrow \forall x \in \mathfrak{I}$, $x \in \mathfrak{p}$. Let $S = \{x, x', x'', \ldots \}$, then $S \cap R = R \times 0$ is non-zero since $\mathfrak{s} \cap S \Rightarrow \exists \mathfrak{p}$ of $R \times 0$ containing $IR^x$ by (2). The preimage of $\mathfrak{p}$ in $R$ is a prime ideal containing $I$ and not meeting $S \Rightarrow x \in \bigcap_{x \in \mathfrak{I}} \mathfrak{p}$

(7). $I \subseteq \mathfrak{I} \Rightarrow \text{V}(I) \supseteq \text{V}(\mathfrak{I})$. Conversely $\mathfrak{p} \supseteq \mathfrak{I} \Rightarrow \mathfrak{p} \cap \bigcap_{x \in \mathfrak{I}} \mathfrak{p} = \mathfrak{I} \Rightarrow \mathfrak{p} \in \text{V}(\mathfrak{I})$

(8). $\mathfrak{p} \supseteq \mathfrak{I} \iff \mathfrak{p} \\supseteq 0$ in $R/I$. The result follows from (10) then

(9). $I(J) \supseteq IJ \supseteq I \supseteq V(I) \cup V(J) \subseteq V(I \cup J) \subseteq V(J)$. Conversely, if $\mathfrak{p} \supseteq IJ$ but not $I \Rightarrow \exists x \in I, x \in \mathfrak{p}$. Moreover, $\forall y \in J, xy \in \mathfrak{p} \Rightarrow y \in \mathfrak{p} \Rightarrow \mathfrak{p} \supseteq J$.

(10). $I \subseteq \Sigma I_a \Rightarrow \text{V}(I) \supseteq \text{V}(I_a) \Rightarrow \bigcap_{a \in A} \text{V}(I) \supseteq \text{V}(\Sigma I_a)$. Conversely $\mathfrak{p} \in \bigcap_{a \in A} \text{V}(I_a) \Rightarrow \mathfrak{p} \supseteq I_a \Rightarrow \bigcap_{a \in A} \mathfrak{p} \supseteq \Sigma I_a$.

(11). It's just a definition.
(12). \( \text{Def} = \emptyset \Leftrightarrow V(f) = \text{Spec}R \Leftrightarrow f \in \bigcap_{\alpha \in \Phi} \sqrt{\alpha} \Leftrightarrow f \text{ nilpotent} \)

(13). \( f = uf' \Rightarrow (f) = (f') \Rightarrow V(f) = V(cf) = V(cf') = V(f') \Rightarrow \text{Def} = \text{Def}' \)

(14). \( \mathfrak{p} \in V(I) \Rightarrow \mathfrak{p} \supseteq I \Rightarrow \exists f \in \mathfrak{p} \text{ st. } f \notin \mathfrak{p} \Rightarrow \mathfrak{p} \in \text{Def} \) and \( \forall g \in \text{Def}, \ g \in V(I) \).

(15). Take complement of (9), where \( I = \langle f \rangle, \ J = \langle g \rangle \). \( \square \)

**Examples:**

1. \( \text{Spec} \mathbb{Z} = \{ (2), (3), (5), \ldots, (0) \} \)

   (0) is not closed, since (0) = V(I) \( \Rightarrow \) (0) \( \supseteq \) I \( \Rightarrow \) I = (0). But V(0) = Spec\( \mathbb{Z} \).

   All other primes are closed.

2. \( \text{Spec} \mathbb{Z}[x]/(x^2 - 4) \)

   We have \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[x]/(x^2 - 4) \rightarrow \text{Spec} \mathbb{Z}[x]/(x^2 - 4) \rightarrow \text{Spec} \mathbb{Z} \), \( \mathfrak{p} \mapsto \mathfrak{p} \cap \mathbb{Z} \)

   **Case I:** \( \mathfrak{p} \cap \mathbb{Z} = (2) \).

   \( \Rightarrow \ 0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2[x]/(x^2 - 4) = \mathbb{Z}[x]/(x^2) \)

   The only prime in \( \mathbb{Z}[x]/(x^2) \) is (x) of multiplicity 2.

   \( \Rightarrow \) The only \( \mathfrak{p} \cap \mathbb{Z} = (2, x) \)

   **Case II:** \( \mathfrak{p} \cap \mathbb{Z} = (p) \), \( p \) odd.

   \( \Rightarrow \ 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p[x]/(x^2 - 4) = \mathbb{Z}/p[x]/((x-2)(x+2)) \)

   The only primes in \( \mathbb{Z}/p[x]/(x^2 - 4) \) are \( (x-2) \) and \( (x+2) \)

   \( \Rightarrow \) The only \( \mathfrak{p} \cap \mathbb{Z} = (p) \) are \( (p, x-2) \), \( (p, x+2) \)

   **Case III:** \( \mathfrak{p} \cap \mathbb{Z} = (0) \)

   \( \Rightarrow \ 0 \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}[x]/(x^2 - 4) \)

   The prime ideals are \( (x-2) \) and \( (x+2) \) in \( \mathbb{Q}[x]/(x^2 - 4) \).

   \( \Rightarrow \) The only primes \( \mathfrak{p} \cap \mathbb{Z} = (0) \) are \( (x-2) \), \( (x+2) \).

3. \( \text{Spec} \mathbb{Z}[x] \)

   Similar as above. Consider \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[x] \), let \( \mathfrak{p} \) be in \( \text{Spec} \mathbb{Z}[x] \)
Case I. \( \mathfrak{p} \cap \mathbb{Z} = (p) \)
\[ \Rightarrow 0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p[x] \]
Since \( \mathbb{Z}/p[x] \) is a P.I.D. \( \Rightarrow \) all primes are of the form \( (\overline{f}) \) where \( \overline{f} \) is an irreducible polynomial of \( \mathbb{Z}/p[x] \) or \( (0) \).
\[ \Rightarrow \mathfrak{p} = (p, \overline{f}) \text{ where } \overline{f} \text{ is a preimage of } \overline{f} \text{ under modulo } p \text{ or } \mathfrak{p} = (p) \]
Case II. \( \mathfrak{p} \cap \mathbb{Z} = (0) \)
\[ \Rightarrow 0 \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}[x] \]
Since \( \mathbb{Q}[x] \) is a P.I.D. \( \Rightarrow \) all primes are of the form \( (\overline{f}) \) where \( \overline{f} \) is a primitive irreducible polynomial in \( \mathbb{Z}[x] \) or \( (0) \)
\[ \Rightarrow \mathfrak{p} = (\overline{f}) \]
4c. \( \text{Spec } k[x, y] \)
This is similar as the case of \( \text{Spec } \mathbb{Z}[x] \), since \( k[x] \), similar as \( \mathbb{Z} \), is a P.I.D.
\( \text{Spec } k[x] = \{ (f) \mid f \text{ an irreducible polynomial } \} \cup \{(0)\}, \) and \( (0) \) is not closed.
Thus similarly as in 3), consider \( 0 \rightarrow k[x] \rightarrow k[x, y] \) and let \( \mathfrak{p} \in \text{Spec } k[x, y] \)
Case I. \( \mathfrak{p} \cap k[x] = (f) \)
\[ \Rightarrow 0 \rightarrow k[x, y] \rightarrow \frac{k[x, y]}{(f)} \]
Since \( \frac{k[x, y]}{(f)} \) is a field, \( k[x, y]/(f) \) is a P.I.D. with ideals of the form \( (\overline{g(x, y)}) \)
where \( g(x, y) \) is a polynomial which mod \( f \) to irreducible in \( \frac{k[x, y]}{(f)} \), and the ideal \( (0) \) of \( \frac{k[x, y]}{(f)} \).
Thus such \( \mathfrak{p}'s \) are \( \{ (f), (g(x, y)), \ldots, \{g(x, y)\} \}
Case II. \( \mathfrak{p} \cap k[x] = (0) \)
\[ \Rightarrow 0 \rightarrow k[x] \rightarrow k[x] \cdot y \]
\( k[x] \cdot y \) is a P.I.D. \( \Rightarrow \) all primes are of the form \( (\overline{g(x, y)}) \) where \( \overline{g(x, y)} \) is an irreducible polynomial in \( k(x, y) \), and the ideal \( (0) \). Thus such \( \mathfrak{p}'s \) are \( \{ (g(x, y)), \ldots, (0) \} \)

\[ \begin{array}{c}
\text{Spec } k[x, y] \\
\text{Spec } k[x] \\
\end{array} \]

\[ \begin{array}{c}
\exists (\overline{f}) \\
\exists (\overline{f'}) \\
\end{array} \]
\[ \exists (0) \text{ : the generic point} \]
\[ \begin{array}{c}
\bullet (\overline{f}, \overline{g}) \\
\bullet (\overline{f'}, \overline{g}) \\
\bullet (\overline{f}, \overline{g'}) \\
\bullet (\overline{f'}, \overline{g'}) \\
\ldots \\
\end{array} \]
\[ \{ \overline{g} \} \text{ : generic points of vertical irreducible sets} \]
\[ \{ \overline{g'} \} \]
\[ \text{horizontal irreducible sets} \]
\[ \begin{array}{c}
\bullet (f, g) \\
\bullet (f', g) \\
\bullet (f, g') \\
\bullet (f', g') \\
\ldots \\
\end{array} \]
\[ \{ (0) \} \]
Lemma:
If \( \varphi : R \to S \) is a ring map, then \( \text{Spec } S \to \text{Spec } R, \beta \mapsto \varphi^{-1}(\beta) \) is continuous and in fact the preimage of standard open sets are standard open sets, i.e.
\[
\text{Spec } (\varphi^{-1}) (\text{Def } f) = \text{Def } (\varphi f), \quad \forall f \in R.
\]
Moreover, if \( R \to R' \to R'' \to \text{Spec } R'' \to \text{Spec } R' \to \text{Spec } R \), and
\[
\text{Spec } \varphi \circ \text{Spec } \varphi' = \text{Spec } (\varphi' \circ \varphi)
\]
i.e. Spec is a contravariant functor.

Lemma:
\( R \to S^tR \to \text{Spec } (S^tR) \to \text{Spec } R \). This is a homeomorphism onto the image, where the image = \{ \beta \in \text{Spec } R | \beta \cap S = \emptyset \}.

Pf: By basic properties of localization, \( \text{Spec } (S^tR) \) is in 1-1 correspondence with primes in \( R \) not meeting \( S \). Moreover, by the previous lemma, the map is continuous. Thus it suffices to check that the map is open. Let \( \text{Def } f \) be an open set in \( \text{Spec } (S^tR) \), then
\[
f = \frac{1}{h}, \quad h \in R, \quad s \in S.
\]
\( f \in S^t \beta \iff \frac{1}{h} = \frac{p}{s} \text{ for some } p \in \beta \iff s^t(s'h-ps) = 0 \text{ for some } p \in \beta \iff sh \in \beta \text{ for some } s \in S \iff h \in \beta \text{ (since } \beta \cap S = \emptyset \). Thus the image of \( \text{Def } f \) is just \( Dch \) and the map is open.

A special case of the lemma: \( S = \{1, f, f^2, \ldots \} \). Then
\[
\text{Spec } (R_f) \cong \text{Spec } (S^tR) \to \text{Spec } R
\]
\( \sim \)
\[\text{Def } f\]

Lemma:
Let \( I \subseteq R \) be an ideal. \( \text{Spec } (R/I) \to \text{Spec } R \) is a homeomorphism onto its image \( V(I) \).

Pf: Similar as above it suffices to check that the map is open. Let \( \text{Def } f \) be an open set of \( \text{Spec } (R/I) \).
\[
\beta \ni (f) \iff \beta \supseteq I \text{ and } \beta \ni (f) \iff \beta \supseteq I + (f) \iff \beta \subseteq V(I) \cap V(f) \text{.}
\]
• Recall some general facts about localization: \( R \) is a ring, \( S \): a multiplicatively
closed subset \( S^{-1}R = \left\{ \frac{r}{s} \mid r \in R, s \in S \right\}/\sim \), where \( \frac{r}{s} \sim \frac{r'}{s'} \iff \exists s'' \in S \) s.t. \( s''(rs'-r's) = 0 \).

Examples:

1. \( S = \{1, f, f^2, f^3, \ldots \} \) for some \( f \in R \)
   
   In this case \( \text{Spec} S^{-1}R \cong \text{Spec} f \subseteq \text{Spec} R \) as a canonical open set.

   e.g. \( R = \mathbb{C}[x], f = x(x-1), R_f = \mathbb{C}[x, \frac{1}{x(x-1)}] \)
   
   \( \text{Spec} \mathbb{C}[x, \frac{1}{x(x-1)}] = \text{Spec} \mathbb{C}[x] \setminus \{(x), (x-1)\} \)

2. \( R = \{ f \in \mathbb{Q}[z] \mid f(0) = f(1) \} \subseteq \mathbb{Q}[z] \)

   Claim: \( R \cong \mathbb{Q}[A, B]/(A^3 - B^3 + AB) \) where \( A \mapsto z^2 - 2, B \mapsto z^3 - z^2 \)

   It's easy to check that \( \mathbb{Q}[A, B] \to R \) is well-defined, and since the image is inside an integral domain, the kernel is a prime ideal \( \mathfrak{p} \) of \( \mathbb{Q}[A, B] \) and certainly contains the element \( A^3 - B^3 + AB \). From example above, we know that \( \text{Spec} \mathbb{Q}[A, B] \) consists of

   four types of elements: \( \{(f(A)), (f(A), g(A, B)) \}
   
   \( (g(A, B)), (0) \}. \) But \( \dim_{\mathbb{Q}} \mathbb{Q}[A, B]/(f(A), g(A, B)) < +\infty \) but \( \dim_{\mathbb{R}} R = +\infty \)

   \( \Rightarrow \mathfrak{p} \) is of the form \( (g(A, B)) \), and since \( A^3 - B^3 + AB \)

   is irreducible, \( \mathfrak{p} = (A^3 - B^3 + AB) \).

   Now pick \( a \in \mathbb{Q} \) with \( a \neq 0, 1, \frac{1}{2} \), and let \( Ra = \{ f \in \mathbb{Q}[z, \frac{1}{z-a}] \mid f(0) = f(1) \} \)

   \( \subseteq \mathbb{Q}[z, \frac{1}{z-a}] \)

   • Claim: \( Ra \) is a finitely generated \( \mathbb{Q} \)-algebra

• Claim: \( \text{Spec} Ra \) maps homeomorphically onto an open subset of \( \text{Spec} R \)

   \( \text{Spec} R \setminus \{ a \text{ closed point} \} \), but \( Ra \) is not isomorphic to \( S^{-1}R \) for any multiplicatively
closed subset of \( R \).
\[
\begin{align*}
\mathbb{Q}[A,B] & \xrightarrow{(A^3-B^2+AB)} R \leftarrow R[z] \\
A & \mapsto z^2 - 2 \\
B & \mapsto z^3 - z^2 \\
\{f \in R[z] \mid f(0) = f(1)\} & \xrightarrow{\cong} R \leftarrow R[z] \\
A & \mapsto z^2 + 2 \\
B & \mapsto z^3 + z^2 \\
A^3 - A^2 + A & = A(z^3 - z^2 + A - z^2 + A)
\end{align*}
\]
Def. A local ring is a ring with only 1 maximal ideal.

Lemma:
Let $R$ be a ring. then TFAE:
(1). $R$ is local
(2). $R$ has a maximal ideal $m$ and all elements in $R \setminus m$ are units.

Pf: (1) $\Rightarrow$ (2). Every non-unit is contained in a maximal ideal, thus in $m$.
(2) $\Rightarrow$ (1). $I$ is a ideal, $n$ contains of non-units $\Rightarrow n \subseteq m$.

Notation: $R$ a ring, $\mathfrak{p}$ a prime ideal, $R_\mathfrak{p} \cong (R \setminus \mathfrak{p})^{-1}R$, called the localization of $R$ at $\mathfrak{p}$.
Fact: $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$.
Note that $\text{Spec } R_\mathfrak{p} = \{ \mathfrak{q} \in \text{Spec } R | \mathfrak{q} \cap (R \setminus \mathfrak{p}) = \emptyset \} = \{ \mathfrak{q} \in \text{Spec } R | \mathfrak{q} \subseteq \mathfrak{p} \}$.

Def. The residue field of $\mathfrak{p}$ is the field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \text{Fraction field of } R/\mathfrak{p}$.

Fundamental diagram: $R \overset{\phi}{\longrightarrow} S$ a ring map, and $\mathfrak{p} \subseteq R$ a prime ideal.
$S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \leftarrow S_\mathfrak{p} \leftarrow S/\mathfrak{p}S \rightarrow (R/\mathfrak{p} \setminus 0)^{-1}S/\mathfrak{p}S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$
\[
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow
\]
$k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \leftarrow R_\mathfrak{p} \leftarrow R \rightarrow R/\mathfrak{p} \rightarrow (R/\mathfrak{p} \setminus 0)^{-1}R/\mathfrak{p} = k(\mathfrak{p})$

Any of the squares induce geometrically a fiber product. The upshot of this diagram is that $\mathfrak{p}$ is in the image of $\text{Spec}(\phi)$: $\text{Spec}(S) \rightarrow \text{Spec}(R)$ iff $k(\mathfrak{p}) \subseteq S \neq 0$ or equivalently $\phi^{-1}(\mathfrak{p}S) = \mathfrak{p}S \neq S_\mathfrak{p}$.

Lemma:
$\text{Spec } R$ is quasi-compact.

Pf: Since $\{ \text{Def}(f_i) \}_{i \in I}$ forms a basis of the Zariski topology, it suffices to show that, if $\text{Spec } R = \bigcup_{i \in I} \text{Def}(f_i)$, then $\exists$ a finite subset $T$ of $I$ s.t. $\text{Spec } R = \bigcup_{i \in T} \text{Def}(f_i)$.
Indeed, $\text{Spec } R = \bigcup_{i \in I} \text{Def}(f_i) \iff \emptyset = \bigcap_{i \in I} \text{V}(f_i) \iff \emptyset = \bigcap_{i \in I} \text{V}(\Sigma_{i \in T}(f_i)) \iff \Sigma_{i \in T}(f_i) = R \iff \exists$ a finite subset of $I$ s.t. $1 = \Sigma_{i \in T} f_i \iff \emptyset = V(\Sigma_{i \in T}(f_i)) \iff \text{Spec } R = \bigcup_{i \in T} \text{Def}(f_i)$.
Example:

\[ R = \prod_{i \in \mathbb{N}} \mathbb{F}_2 \xrightarrow{p} \mathbb{F}_2 \Rightarrow \ker p_i \text{ is a maximal ideal (prime). Note that } \forall x \in R, x^2 = x \Rightarrow \forall \beta \text{ prime, } R/\beta \approx \mathbb{F}_2. \text{ But are } \ker p_i \text{ s all the prime ideals in } R? \]

If yes \( \Rightarrow \text{Spec } R = \{ \ker p_i \} \text{ is a countable set with discrete topology, which cannot be quasicompact, contradiction with the lemma. } \Rightarrow \exists \text{ "funny" primes. One such prime ideal can be seen as follows: } \oplus \mathbb{F}_2 \hookrightarrow \prod \mathbb{F}_2 \text{ as an ideal, which must be contained in a maximal ideal, any such a maximal ideal is not a } \ker p_i. \]

Def: Let \( X \) be a topological space

1) \( X \) is connected iff \( (X = T_1 \cup T_2, T_i \text{ both open and closed } \Rightarrow T_1 \text{ or } T_2 \text{ is } \emptyset) \).

2) We say \( T \) is a connected component of \( X \) if it is a maximal connected subset.

Lemma.

\( X \) a topological space, \( T \subseteq X \) is connected \( \Rightarrow \overline{T} \) is connected. Every \( x \in X \) is contained in a connected component. Connected components are always closed but not necessarily open. \( \square \)

Example: a closed connected component which is not open

A sequence of points converge to a point \( p \), then \( \{ p \} \) is a connected component but not open.

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Def: An idempotent of a ring is an element \( e \in R \) satisfying \( e^2 = e \). Trivial idempotents are \( 1, 0 \), (which are the only ones in a domain).

If \( e \) is an idempotent, so is \( 1 - e \). The standard open set \( D(e) = \{ \beta \in \text{Spec } R \mid e \notin \beta \} \) = \{ \beta \in \text{Spec } R \mid \overline{e} = 0 \text{ in } R/\beta \} = \{ \beta \in \text{Spec } R \mid \overline{e} = 1 \in R/\beta \subseteq \mathbb{K}(\beta) \}

= \{ \beta \in \text{Spec } R \mid e \rightarrow 1 \in \mathbb{K}(\beta) \}

Similarly, \( D(1-e) = \{ \beta \in \text{Spec } R \mid e \rightarrow 0 \in \mathbb{K}(\beta) \} \).
Since 1, 0 are the only possible images in \( k(fp) \) for our idempotents, we conclude that 
\[ \text{Spec}(R) = \text{D}(e) \sqcup \text{D}(1-e). \]

**Lemma:**
The construction \( e \mapsto \text{D}(e) \) gives a bijection between:
- Idempotents of \( R \)
- Open and closed subsets of \( \text{Spec}(R) \)

**Upshot:** \( \text{Spec}(R) \) is connected iff \( R \) has only trivial idempotents.

To prove the lemma, we first show:

**Lemma:**
Given a ring \( R \) and elements \( f_1, \ldots, f_r \) s.t. \( \langle f_1, \ldots, f_r \rangle = R \), then the sequence
\[ \begin{array}{c}
o \\ \to R \\ \alpha \downarrow \\ \bigoplus Rf_i \\ \beta \downarrow \\ \bigoplus R_{f_if_j} \\ \end{array} \]

is exact, where \( \alpha(f_i) = (f_1, \ldots, f_r) \) \( \beta(x_i, \ldots, x_n) = (x_i - x_j); i, j \)

Pf: \( \alpha(f_i) = 0 \Rightarrow f_i f_j = 0 \) \( i = 1, \ldots, r \), and we may take a common \( k \) s.t. \( f_i^k f_j = 0 \)
But \( (f_1^k, \ldots, f_r^k) = R \Rightarrow i = \sum a_i f_i^k \Rightarrow f = \sum a_i f_i = 0 \)

Obviously, \( \beta \alpha = 0 \). Moreover, if \( \beta \left( \frac{a_1}{f_1}, \ldots, \frac{a_r}{f_r} \right) = 0 \), i.e. \( (a_i f_i^k - a_j f_j^k) f_i f_j = 0 \) \( \forall i, j \).

Since there are only finitely many terms involved, we may take a common \( N \)
and common \( k \) s.t. \( (a_i f_i^k - a_j f_j^k) f_i f_j = 0 \) \( \Rightarrow a_i f_i^k + a_j f_j^k = 0 \).

Replace \( a_i \) by \( a_i f_i^N \)
\( f_i \) by \( f_i^{k+N} \), we may assume the element \( 0 \) given by \( \left( \frac{a_1}{f_1}, \ldots, \frac{a_r}{f_r} \right) \) and \( f_i a_j = f_j a_i \).

Furthermore \( \langle f_1, \ldots, f_r \rangle = 1 \) \( \Rightarrow \exists b_j \) s.t. \( \Sigma b_j f_j = 1 \). Now take \( g = \Sigma b_j a_j \in R \).

Lemma: \( f_i g = \Sigma b_j a_j f_i = \Sigma b_j a_i f_j = (\Sigma b_j f_j) a_i = a_i \Rightarrow g \mapsto \frac{a_i}{f_i} \) in \( R_{f_i} \).

Proof of the lemma.
If \( V \) is an open and closed set, so is \( V^c \), thus we may cover each of them by
finitely many standard open sets, say \( V = \bigcup_i \text{D}(f_i) \) \(, V^c = \bigcup_j \text{D}(h_j) \).

Since \( V \) \(\cup\) \( V^c = \text{Spec}(R) \Rightarrow \langle f_1, \ldots, f_r, h_1, \ldots, h_s \rangle = R \).

Consider \( o \mapsto R \xrightarrow{\alpha} \bigoplus R_{f_i} \xrightarrow{\beta} \bigoplus R_{f_i f_j} \xrightarrow{\delta} \bigoplus R_{f_i h_j} \).

Note \( R_{f_i h_j} = 0 \).

Let \( \bar{a} \) be the element \( (1, \ldots, 1, 0, \ldots, 0) \in \bigoplus R_{f_i} \xrightarrow{\delta} \bigoplus R_{f_i h_j} \).
Then $\beta(e) = 0$, thus by the previous lemma, $\bar{e} = \alpha(e)$ for a unique $e$, and $e$ maps to 1 in every $\mathfrak{k}_p$ for $p \in \bigcup_{i=1}^{s} \mathfrak{D}(i)$ and 0 in $\mathfrak{k}_p$ for every $p \in \bigcup_{j=1}^{s} \mathfrak{D}(j)$.
Moreover $\alpha(e^2) = \bar{e}^2 = \bar{e} = \alpha(e)$ and by injectivity of $\alpha$, $e^2 = e$, thus $e$ is an idempotent.

The uniqueness of $e$ also follows from the injectivity of $\alpha$.

\[\Box\]

Def: Let $X$ be a topological space.

(a) We say $X$ is irreducible if $(X = \mathbb{Z}_1 \cup \mathbb{Z}_2$, $\mathbb{Z}_1$ closed in $X \Rightarrow \mathbb{Z}_1 = X$ or $\mathbb{Z}_2 = X$).

(b) An irreducible component of $X$ is a maximal irreducible subset of $X$.

Lemma:
Let $X$ be a topological space. $T \subseteq X$ irreducible $\Rightarrow \overline{T}$ is irreducible. Irreducible components are closed and $\forall x \in X$ is contained in an irreducible component of $X$.

\[\Box\]

Examples:

(a) Connected components and irreducible components.

One connected component

Two irreducible components

Two connected components

Two irreducible components

(b) A singleton set is irreducible $\Rightarrow x \in X$, $\overline{\{x\}}$ is irreducible.

Def: Let $X$ be a topological space.

(a) Let $Z \subseteq X$ be closed, irreducible. A generic point of $Z$ is a point $z \in X$ s.t. $\overline{\{z\}} = Z$

(b) The space $X$ is called sober if every irreducible closed subset has a unique generic point.
Example:

Suppose \( R \) is a domain. Then \( \{0\} \in \text{Spec}R \) and \( \overline{\{0\}} = \text{Spec}R \).

It follows that \( \text{Spec}R \) is irreducible and \( \{0\} \) is the generic point.

Lemma:

Let \( R \) be a ring, the irreducible closed subsets of \( \text{Spec}R \) are exactly the subsets \( V(\mathfrak{p}) \) with \( \mathfrak{p} \) prime. The irreducible components of \( \text{Spec}R \) are the subsets \( V(\mathfrak{p}) \) with \( \mathfrak{p} \) minimal prime.

Upshot: \( \text{Spec}R \) is a sober topological space.

Recall that \( V(I) = V(\sqrt{I}) \), \( \sqrt{I} = \cap_{\mathfrak{p} \in P} \mathfrak{p} \Rightarrow \exists I \) correspondence between

1. closed subsets of \( \text{Spec}R \)
2. radical ideals of \( R \)

\[
\begin{align*}
\mathcal{Z} & \quad \mapsto \quad \mathfrak{n} \\
V(I) & \quad \mapsto \quad \sqrt{I}
\end{align*}
\]

Proof of lemma: It suffices to consider radical ideals. Let \( I \) be radical. \( V(I) \) irreducible \( \Leftrightarrow \) \( I \) is a prime.

If \( I \) is not a prime \( \Rightarrow \exists a, b \notin I \), \( ab \in I \). \( \Rightarrow V(I) = V(I+(a)) \cup V(I+(b)) = (V(I) \cap V(I+(a))) \cup (V(I) \cap V(I+(b))) \) but \( V(a) \cap V(I) \neq V(I) \), \( V(b) \cap V(I) \neq V(I) \).

Since \( a, b \notin I \). Hence \( V(I) \) is not irreducible.

Conversely if \( I \) is prime. \( V(I) = \overline{\{I\}} \)

Examples:

1. \( \text{Spec}(\mathbb{Z}[x]/(x^2-4)) \) revisited.

\( \text{Spec}(\mathbb{Z}[x]/(x^2-4)) \overset{\pi}{\longrightarrow} \text{Spec}\mathbb{Z} \) is given as below

There are 2 irreducible components, corresponding to two minimal primes i.e. \( \overline{\{(x-2)\}} \) and \( \overline{\{(x+2)\}} \).
(a) $k$: a field. What are the irreducible components of $\text{Spec}(k[x,y]/(xy))$? (or what are the irreducible components of $V(xy)$?)

$\beta \supseteq (xy) \Rightarrow xy \in \beta \Rightarrow \text{either } x \in \beta \text{ or } y \in \beta. \Rightarrow \beta \supseteq (x) \text{ or } \beta \supseteq (y)$

Thus $(x), (y)$ are the minimal primes, corresponding to the irreducible components $V(xy) = x$-axis $\cup$ $y$-axis.

(b) As a generalization, what are the irreducible components of $\text{Spec} R$, where $R = k[x_1, x_2, x_3, y_1, y_2, y_3] / I$, and $I$ is generated by the entries of the $2 \times 2$ matrix $X \cdot Y = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$?

To find the irreducible components, we shall use a property of $R$ to be introduced later, that it is Jacobson. Then to find its irreducible components, it suffices to find the irreducible components of $\text{max Spec } R$.

Define a $GL(2,k) \times GL(2,k) \times GL(2,k)$ action on $X \cdot Y$ by $g \cdot X \cdot g^{-1} \cdot Y \cdot g$. Then since $GL(2,k) \times GL(2,k) \times GL(2,k)$ is irreducible, so is its image in $M^4 \times M^4$. In particular, this action preserves $\{X \cdot Y \mid X \cdot Y = 0\}$. It suffices to classify the orbits of this set under the group action.

Now, by simple linear algebra, we know that the orbits of $GL(2,k) \times GL(2,k)$ on $M^2(k)$ by $g \cdot X \cdot g^{-1}$ are $\{(0,0), (0,0), (0,0)\}$. Now we discuss case by case:

a. $g \cdot X \cdot g^{-1} = (0,0)$. Then $0 = g \cdot X \cdot g^{-1} \cdot Y \cdot g = (g^{-1} \cdot Y \cdot g) \cdot g = Y \cdot g$. This gives us the irreducible component $Y = 0$, or the ideal $(y_1, y_2, y_3, y_4)$. Since $\{X = g \cdot X \cdot g^{-1} \mid g \in GL(2,k)\} \cong GL(2,k)$ is Zariski dense in $M^2(k)$.

b. $g \cdot X \cdot g^{-1} = (0,0)$. Then $0 = X \cdot Y$ arbitrary. Hence the ideal is $(x_1, x_2, x_3, x_4)$.

c. $g \cdot X \cdot g^{-1} = (0,0)$. In this case more care is needed. Let's call this component(s) $Z$, and we will show that $Z$ is actually irreducible. To do this, it suffices to show that $Z \setminus \{0\}$ is irreducible, where $0$ is the closed point $(x_1, y_1)$. We will show that $Z \setminus \{0\}$ is covered by 2 open neighborhoods $U, V$ s.t. $U \cdot V, U \cdot V$ are irreducible, the result then follows.
8.2 Images and Ring Maps of Finite Presentation.

Some topology:

Def: (a) We say a topological space is quasi-compact iff every open covering has a finite refinement.
(b) A continuous map \( f : X \to Y \) is called quasi-compact iff \( f^{-1}(V) \) is quasi-compact for every open quasi-compact \( V \) in \( Y \).
(c) We say that \( Z \hookrightarrow X \) is retro-compact iff the map \( Z \hookrightarrow X \) is quasi-compact.

Remark: Often a map \( f : X \to Y \) is called proper iff \( f^{-1}(K) \) is compact whenever \( K \) is compact. (In particular, in Ti spaces, points are compact, thus fibers are compact if \( f \) is proper.) Note the difference in requiring \( K \).

Def: \( f : X \to Y \) map of topological spaces.
(a) \( f \) is closed iff the image of every closed set is closed.
(b) \( f \) is universally closed iff \( \forall \) topological space \( Z \), the map \( Z \times X \to Z \times Y \) is closed.
(c) \( f \) is quasi-proper iff \( f^{-1}(K) \) is quasi-compact whenever \( K \subseteq Y \) is quasi-compact.

Fact: quasi-proper \( \iff \) universally closed \( \iff \) \( f \) is closed and \( \forall Y \subseteq Y \), \( f^{-1}(Y) \) is quasi-compact.

Ex. (i). Prove or find reference for the fact

(1). Explain what this means for locally compact Hausdorff spaces.
(2). What’s the relation between (b) and the condition “\( \forall X \to Y, Z \to Y \), the induced map \( X \times Y \to Z \) is closed”? 
Def. A subset $E \subseteq X$ is called constructible if it's a finite union of subsets of the form $U \cap V^c$ where $U, V \subseteq X$ are open and retrocompact.

Lemma: The collection of constructible subsets is closed under finite intersections.

Proof: Suppose $E$ is constructible, $E = \bigcap_{i=1}^{n} U_i \cap V_i^c.$

Then $E^c = \bigcap_{i=1}^{n} (U_i \cup V_i) = \bigcup_{k=0}^{n} \bigcap_{1 \leq i_1 < i_2 < \cdots < i_k \leq n, 1 \leq j_1 < \cdots < j_n \leq n} (U_{i_1} \cup \cdots \cup U_{i_k}) \cap (V_{j_1} \cap \cdots \cap V_{j_n})$

$\bigcup_{i=1}^{n} U_i \cap V_i = \bigcup_{i=1}^{n} U_i \cap V_i^c \cup \bigcap_{i=1}^{n} U_i \cap V_i = \bigcup_{i=1}^{n} U_i \cap V_i^c$

Thus it suffices to show that finite union and intersection of retrocompact sets are still retrocompact. Take $W$ quasi-compact open in $X.$

$(U \cup V) \cap W = U \cap (V \cap W)$ quasi-compact (q.c.) since $V$ retro-compact (r.c.)

$= U \cap (q.c.) = (q.c.)$

$(U \cup V) \cap W = (U \cap W) \cup (V \cap W) = (q.c.) \cup (q.c.) = (q.c.)$ \hfill $\square$

Example: (Back to commutative algebra)

$\mathbb{C}[x,y] \rightarrow \mathbb{C}[u,y] \xrightarrow{x \mapsto uy \quad y \mapsto y}$

What's the image?

$\mathbb{C}^2 \rightarrow \mathbb{C}^2$

$(uy, y) \leftrightarrow (u, y)$

$\Rightarrow$ image doesn't contain $y=0$ except $(0, 0)$

$\Rightarrow$ image $= \text{D}(y) \cup \{\text{the maximal ideal } (x,y)\}$

open (q.c.) closed $= \text{open}^c$, and in Noetherian spaces open sets are q.c.

Lemma:

$R$: a ring, $U \subseteq \text{Spec } R$ open. TFAE:

(a). $U$ retro-compact

(b). $U$ is quasi-compact

(c). $U$ is a finite union of standard open sets.
Pf: (a) ⇒ (b) : Spec R is q.c.
   (b) ⇒ (c) : Any open set is the union of standard open sets
   (c) ⇒ (a) : Enough to show that \( \text{D}(f_j) \subseteq \text{Spec } R \) is r.c.

Take \( W \) q.c. by (b) ⇒ (c), \( W = \bigcup_{j=1}^n \text{D}(g_j) \Rightarrow \text{D}(f) \cap W = \bigcap_{j=1}^n \text{D}(g_j) = \bigcap_{j=1}^n \text{Spec } R_{g_j} \).
Thus finite union of q.c. is still q.c. \( \square \)

Examples:
\( R = \mathbb{C}[x_1, \ldots, x_n, \ldots] \), \( U = \bigcup_{j=1}^\infty \text{D}(x_j) \) is not q.c.
\( U = \text{Spec } R \setminus \{(x_1, \ldots, x_n, \ldots)\} \)

Take the open cover of \( \{\text{D}(x_j)\} \) itself, then the point \( (0, 0, \ldots, 1, 0, 0, \ldots) \)

is only in \( U_j \) but not any other \( U_i \) (\( i \neq j \)). Thus this cover has no finite subcover.

This is because there is no finitely generated ideal whose radical is the maximal ideal \( (x_1, \ldots, x_n, \ldots) \), which is the content of the next:

Lemma: \( \text{Spec } R \setminus V(I) \) is not quasi-compact ⇔ there do not exist finitely many
\( f_1, \ldots, f_r \in R \) s.t. \( \sqrt{(f_1, \ldots, f_r)} = \sqrt{I} \).

We can't change the condition in the lemma to "I being finitely generated"

since we have the counter-example:

Example: (SpecR Noetherian ⊄ R Noetherian.)
\( \mathbb{C}[x_1, \ldots, x_n, \ldots] / (x_1, x_2, x_3, \ldots) \) \( I = (\bar{x}_1, \bar{x}_2, \ldots) \) \( \sqrt{I} = \sqrt{\bar{I}} \) and \( I \)

is not finitely generated, yet the spectrum is a singleton thus q.c.

Pf of lemma:
\( \text{Spec } R \setminus V(I) \) not q.c. If there were such \( f_1, \ldots, f_r \Rightarrow V(f_1, \ldots, f_r) = V(I) \)

\( \Rightarrow \bigcup_{i=1}^r \text{D}(g_i) = \text{Spec } R \setminus V(I) \) but the L.H.S. is q.c.

Conversely if \( \text{Spec } R \setminus V(I) \) q.c. ⇒ it's the union of finitely many \( \text{D}(g_i) \) \( i = 1, \ldots, r \)
\( \Rightarrow V(g_1, \ldots, g_r) = V(I) \Rightarrow \sqrt{(g_1, \ldots, g_r)} = \sqrt{I} \). \( \square \)
Lemma:
\( R \to S \) a map of rings, then \( \text{Spec } S \to \text{Spec } R \) is a quasi-compact map of topological spaces.

Proof: This is because the preimage of any \( D(f), f \in R \) is the standard open set \( D(q(f)) \).

Lemma:
Let \( R \) be a ring and let \( T \subseteq \text{Spec } R \) be a constructible set, then there exists a finitely presented \( R \)-algebra \( S \) s.t. the image of \( \text{Spec } S \to \text{Spec } R \) is \( T \).

Proof: It suffices to write \( U \cap V = (\bigcap_{j=1}^{m} D(f_j)) \cap V(q_1, \ldots, q_r) \) as the image of \( \text{Spec } S \to \text{Spec } R \) for some finitely presented \( S \).

Now, \( U \cap V = \bigcap_{j=1}^{m} (D(f_j) \cap V(q_1, \ldots, q_m)) = \bigcap_{j=1}^{m} \text{Spec } (Rf_j / (q_1, \ldots, q_m) Rf_j) \)

Thus we may take \( S = \bigoplus_{j=1}^{m} (R / (q_1, \ldots, q_m) f_j) = \bigoplus_{j=1}^{m} R[x_j / (x_j f_j - q_1, \ldots, q_m)] \), which is a finitely presented \( R \)-algebra.

Lemma:
\( R \) a ring, \( f \in R \). Then the image in \( \text{Spec } R \) of a constructible set in \( \text{Spec } Rf \) is constructible.

Proof: Let \( U, V \) be quasi-compact open and \( T = U \cap V^c \), we have to show that the image of \( T \) in \( \text{Spec } Rf \) is constructible.

Now \( \text{Spec } Rf \to \text{Spec } R \), \( U, V \to U', V' \) (homeomorphism)

\[
\begin{align*}
U, V & \cong U', V' \\
D(f) & \Rightarrow \text{im } T = U' \cap V^c
\end{align*}
\]

It suffices to show that \( U', V' \) are retrocompact in \( \text{Spec } R \)
or equivalently \( U', V \) are a finite union of standard open sets. But this is clear since standard open sets in \( \text{Spec } Rf \) is a subclass of standard open sets of \( \text{Spec } R \).

Lemma:
\( R \) a ring, \( I \subseteq R \) is a finitely generated ideal. Then the image of any constructible set of \( \text{Spec } R/I \) is constructible in \( \text{Spec } R \).
Pf: Let \( I = (f_1, \ldots, f_m) \), then \( \text{V}(I) = \bigcap_{i=1}^m \text{V}(f_i) \cong \text{Spec} R/I \).

Any constructible set \( T \) in \( \text{Spec} R/I \) is of the form \( U \cap V^c \), \( U, V \) quasi-compact open. It suffices to show that \( U = U' \cap \text{V}(I) \), \( V = V' \cap \text{V}(I) \), where \( U', V' \) are quasi-compact open subsets of \( \text{Spec} R \), then

\[
\text{int}(T) = (U' \cap \text{V}(I)) \cap (V' \cap \text{V}(I))^c = U' \cap \text{V}(I) \cap (V' \cup \text{V}(I))^c
\]

\[
= U' \cap (V' \cup \bigcup_{j=1}^m D(f_j))^c \quad \text{and} \quad V' \cup \bigcup_{j=1}^m D(f_j) \text{ is quasi-compact.}
\]

Indeed, if \( U = \bigcup_{a \in R/I} D(\overline{g}_a) \) where \( \overline{g}_a \in R/I \Rightarrow \) we may take \( U' = \bigcup_{a \in R/I} D(\overline{g}_a) \) where \( \overline{g}_a \) is a lifting of \( \overline{g}_a \) in \( R \).

\[\square\]

\textbf{Lemma:}

R a ring, then the map \( \text{Spec} R \times \mathbb{A} \rightarrow \text{Spec} R \) is open, and the image of any standard open is quasi-compact open.

Pf: The second statement implies the first.

Take \( f \in R \times \mathbb{A} \), \( f = a_0x^n + \cdots + a_n \), then we have the diagram:

\[
R \times \mathbb{A} \xrightarrow{f} k[\mathbb{A}] \xrightarrow{\text{Spec} \phi} \text{Spec} R
\]

Thus, \( \phi \) is the image of \( \text{Spec} f \) \( \Rightarrow k[\mathbb{A}] \times \mathbb{A} \) is non-zero.

\[
\Rightarrow \overline{f} \text{ is not } 0 \text{ in } k[\mathbb{A}] \times \mathbb{A} \text{ since } k[\mathbb{A}] \times \mathbb{A} \text{ is a domain}
\]

\[
\Rightarrow \text{The coefficients of } f \text{ mod } \beta \text{ are not all } 0
\]

\[
\Rightarrow \beta \not\in (a_0, \ldots, a_n)
\]

\[
\Rightarrow \beta \in \bigcup_{i=1}^m D(a_i)
\]

Thus \( \text{imD}(f) = \bigcup_{i=1}^m D(a_i) \), which is quasi-compact open in \( \text{Spec} R \).

\[\square\]

\textbf{Lemma:}

Let \( R \) be a ring, \( f, g \in R \times \mathbb{A} \). Assume the leading coefficient of \( g \) is a unit, then there exist \( r_i \in R \), \( i = 0, \ldots, d-1 \), s.t. \( \text{Im}(D(f) \cap V(g)) = \bigcup_{i=0}^{d-1} D(r_i) \) under the map \( \text{Spec} R \times \mathbb{A} \rightarrow \text{Spec} R \).

Pf: Write \( g(x) = ux^d + a_{d-1}x^{d-1} + \cdots + a_0 \), and consider \( A = R \times \mathbb{A} / (g) \), since \( u \) is a unit, \( A \cong R^d \) as \( R \)-modules.
Def \cap V(\mathfrak{g}) = \text{Spec}(\overline{A}_{\mathfrak{g}}) \text{ where } \overline{f} = f \mod(\mathfrak{g}).

By Euclidean algorithm, we may assume \deg f < \deg(\mathfrak{g}) \text{, and consider the } R \text{ linear map of } A: A \to A, \overline{\alpha} \mapsto f \cdot \overline{\alpha}. \text{ Then this map has a characteristic polynomial } P(T) \text{ and } P(f) = f^d + r_{d-1} f^{d-1} + \cdots + r_0 = 0. \text{ Consider the diagram:}

$$
\begin{array}{ccc}
A_{\overline{f}} & \longrightarrow & k \otimes_{\mathfrak{g}} A_{\overline{f}} = (\frac{k(\overline{f}) \otimes \mathfrak{g}}{\mathfrak{g}^2})_{\overline{f}} \\
\uparrow & & \uparrow \\
R & \longrightarrow & k \otimes_{\mathfrak{g}} 
\end{array}
$$

Now, \( g \in \text{Im}(D_{\overline{f}} \cap V(\mathfrak{g})) \iff \frac{k(\overline{f}) \otimes \mathfrak{g}}{\mathfrak{g}^2} \neq 0 \iff \overline{f} \text{ is not nilpotent in } \frac{k(\overline{f}) \otimes \mathfrak{g}}{\mathfrak{g}^2} \iff \text{ The characteristic polynomial of } (u \mapsto \overline{f} \cdot u) \text{ is not } 0 \text{ in } k(\overline{f}) \otimes \mathfrak{g} / \mathfrak{g} \cong k \otimes_{\mathfrak{g}} \mathfrak{g}^d \iff P(T) \neq T^d \mod{\mathfrak{g}} \iff \mathfrak{g} \neq \langle r_0, \ldots, r_{d-1} \rangle \iff \mathfrak{g} \in \bigcup_{i=0}^{d-1} D(f_i) \hfill \square

Geometrically:

![Diagram showing geometric interpretation]

\[ \Delta : \mathfrak{g}(x) = 0 \quad \Rightarrow : \mathfrak{g}(x) \neq 0 \]

If \( f \) passes through all points of \( F \cap V(\mathfrak{g}) \) where \( F \) is a fiber, then \( \pi(F) \in \text{im.} \)

Thm: \( \text{(Chevalley)} \)

Assume \( R \twoheadrightarrow S \) is of finite presentation, then \( \text{Spec } S \to \text{Spec } R \) preserves constructible sets.

Pf: \( \text{We do reduction of the ring map. } S = R[x_1, \ldots, x_n] / (f_1, \ldots, f_m) \) We have shown that \( \text{Spec } (\phi) \) preserves constructible sets, it thus suffices to show that \( \text{Spec } (\psi) \) preserves constructible sets.

Furthermore, we are reduced to prove that the image of \( U \cap V^c \) is constructible under \( \psi : \text{Spec } R[x] \to \text{Spec } R \) where \( U = \bigcap_i D(f_i) \) \( V = \bigcup_j D(g_j) \) \( f_i, g_j \in R[x] \)
\[ \bigcup i \in \mathbb{N} V(f_i) \cap V(g_1, \ldots, g_m) \] = \bigcup i \in \mathbb{N} (D(f_i) \cap V(g_1, \ldots, g_m)). \] Hence it suffices to prove for 
\[ D(f_i) \cap V(g_1, \ldots, g_m) = E. \]

Now suppose \( c \in R \). Then we have:
\[
\begin{array}{c}
E \subseteq \text{Spec} R[c] = \text{Spec} R[c[c] \sqcup \text{Spec} \frac{R[c]}{cR[c]} \\
\downarrow \pi_i \downarrow \pi_i \downarrow \pi_i \\
\text{Spec} R = \text{Spec} R[c] \sqcup \text{Spec} \frac{R[c]}{cR[c]}
\end{array}
\]

If \( E \) is constructible, let \( E_1 = E \cap \text{Spec} R[c[c], \ E_2 = E \cap \text{Spec} \frac{R[c]}{cR[c]} \). It suffices to show that \( \pi_i(E_1) \) and \( \pi_i(E_2) \) are constructible, since by previous lemmas, closed and open immersions preserve constructible sets. Note that:
\[
E_1 = D(f'_{1}) \cap \bigvee V(g_{i+1n}^{-}, \ldots, g_{m}), \text{ where } f_{i+1n} \text{ are the images of } f, g_{i} \text{ in } R[c[c]; \\
E_2 = D(f'_{1}) \cap \bigvee V(g_{i+1n}^{-}, \ldots, g_{m}), \text{ where } f_{i+1n} \text{ are the images of } f, g_{i} \text{ in } \frac{R[c]}{cR[c]}.
\]

Number \( g_{i} \)'s so that \( \deg g_{i} \leq \deg g_{i+1} \). We prove by induction on \( m + \sum \deg g_{i} = N \), \( N = 1 \): \( m = 1, \deg g_{1} = 0 \). Take \( R[c] = g_{1} \). \( E_1 = \emptyset, E_2 = D(f'_{1}) \), the result follows from lemma for closed immersions.

Suppose the result is true for \( N < k \), \( N = k \): let \( c \) be the leading coefficient of \( g_{1} \). Then \( \pi_i(E_2) \) is constructible since now \( \deg g_{1} < \deg g_{i+1} \Rightarrow \sum \deg g_{i} + m < N. \)

For \( \pi_i(E_1) \), now \( g_{1} \) is a polynomial with leading coefficient a unit. Thus we may either write \( g_{m} = h g_{1} + r \) with \( \deg r < \deg g_{1} \), or \( g_{m} = h g_{1} \) so \( E_2 = D(f'_{1}) \cap \bigvee V(g_{i+1n}^{-}, \ldots, g_{m}) \).

In any case \( N \) decreases and by induction hypothesis we are done. \( \square \)

Example: (What does this mean in number theory?)

\[ 0 \longrightarrow \mathbb{Z} \longrightarrow R : \text{finite type.} \]

Chevalley's thm \( \Rightarrow \text{ImSpec} \mathbb{R} \) in \( \text{Spec} \mathbb{Z} \) is constructible. But the sets of the form \( \bigcup \mathbb{N} \mathbb{Z} \) in \( \text{Spec} \mathbb{Z} \) are either (i) finite set of closed points or (ii) non-empty open subsets of \( \text{Spec} \mathbb{Z} \).

Claim: \( \text{ImSpec} \mathbb{Z} \) is not a finite closed set.

Indeed, otherwise \( V(n, 1) = \text{Spec} R \), i.e. \( n \) is nilpotent in \( R \), but this is impossible since we assumed that \( \mathbb{Z} \) injects into \( R \).

Conclusion: \( \forall n \) all but finitely many primes \( p \) in \( \mathbb{Z} \), \( \exists q \in \text{Spec} R \) s.t. \( q \cap \mathbb{Z} = p \).
§3. Noetherian Rings

Basic fact: $R$ Noetherian

$\iff$ Every ideal $I$ of $R$ is finitely generated.

$\iff$ Ascending chain condition (a.c.c.) for ideals of $R$.

Lemma:

Any finitely generated ring over a Noetherian ring is Noetherian.

Pf: Let $R$ be Noetherian. The following two facts follow from the relation between the set of ideals in $R$ and the set of ideals in $R/I$ and $S^{-1}R$.

(i). If $I \subseteq R$ is an ideal, then $R/I$ is Noetherian.

(ii). If $S$ is a multiplicative set, then $S^{-1}R$ is Noetherian.

Along with the next fact, these facts together prove the lemma.

(iii) $R[x]$ is Noetherian.

Suppose $J_1 \subseteq J_2 \subseteq \ldots$ is a chain of ideals in $R[x]$. Set $I_i.d = \{0\} \cup \{\text{leading coefficients of polynomials of deg } d \text{ in } J_i\}$, which is obviously an ideal. Moreover, we have $i \leq i', d \leq d' \Rightarrow I_i.d \subseteq I_{i'}d'$.

Claim: there are at most finitely many distinct ideals $I_i.d$ among the collection of ideals $\{I_i.d \mid i, d \in \mathbb{N}\}$.

In fact, for each fixed $i$, $\{I_i.1 \subseteq I_i.2 \subseteq \ldots\}$ so by the Noetherian hypothesis, exists maximal element $I_i.d$ such that whenever $d' \geq d \Rightarrow I_i.d' = I_i.d$. Moreover, we have $I_i.d \subseteq I_{i+1}.d_{i+1}$, since $I_{i+1}.d_{i+1} = I_{i+1}.d_{i+1} + b, b \neq 0$ and we may choose such $b$ that $d_{i+1} + b > d_i$.

Again by the Noetherian hypothesis, exists maximal element in the chain $\{I_i.d \subseteq I_i.d_2 \subseteq \ldots\}$, say $I_i.d_{N_i}$, and then we conclude that there are only finitely many distinct ones for $d > d_{N_i}$. Furthermore, below $d = d_{N_i}$, each horizontal chain is bounded eventually and thus with finitely many distinct ones. Now take $I_{N_i}.d_N$ so that all distinct types of ideals lie on the lower left hand side of it.

Claim: $J_N = J_{N+1} = \ldots$

If $f \in J_N$, $M > N$, we argue by induction on $\deg f$ that $f \in J_N$.

$\deg f = 0 \Rightarrow f = r \in I_{N,0} = I_{N,0} \Rightarrow r \in J_N$. 
Now suppose we have dealt with all cases where \( \deg f < k \).

Now \( \deg f = k \). Since \( N \) is so chosen we know that \( \text{I.m.}_k = \text{I.n.}_k \), and thus \( \exists g \in J_n \), \( \deg g = k \) and \( g \) has the same leading coefficient as \( f \). \( \Rightarrow f - g \in J_m \) has degree \( < k \).
By induction hypothesis, \( f - g \in J_n \) \( \Rightarrow f \in J_n \). \( \square \)

Since any field or principle ideal domain is Noetherian, we have:

Cor.

Any finite type algebra over a field \( k \) or \( \mathbb{Z} \) is Noetherian. \( \square \)

Cor.

Any finite type algebra over a Noetherian ring \( R \) is of finite presentation over \( R \).

Pf: Since \( R[x_1, \ldots, x_n] \) is also Noetherian, any ideal of it is finitely generated. \( \square \)

Def: A topological space \( X \) is Noetherian if the descending chain condition (d.c.c.) holds for all closed subsets of \( X \).

Lemma:

If \( R \) is a Noetherian ring, then \( \text{Spec} R \) is a Noetherian topological space. \( \square \)

The converse is not true:

Example: A Noetherian topological space whose ring is not Noetherian.

Take \( R = \mathbb{k}[x_1, x_2, \ldots] / (x_1, x_2, x_3, \ldots) \). Then \( \text{Spec} R = \{ \text{pt} \} \) and thus Noetherian.

But \( R \) is not since \( \mathfrak{m} = (x_1, x_2, \ldots) \) is not finitely generated.

Lemma:

Let \( X \) be a noetherian topological space.

(i). Any \( T \subseteq X \) is Noetherian.

(ii). \( X \) has finitely many irreducible components.

(iii). Any irreducible component of \( X \) contains a non-empty open subset of \( X \).

Pf: (i). Any descending chain of closed subsets is of the form \( \mathbb{Z}_1 \cap \mathbb{Z}_2 \cap \mathbb{Z}_3 \cap \ldots \) where \( \mathbb{Z}_i \)'s are closed subsets of \( X \). Then we can form a descending chain of closed subsets of \( X \) with the same intersection as \( \mathbb{Z}_1 \cap \mathbb{Z}_2 \), namely \( \mathbb{Z}_1 \supseteq \mathbb{Z}_1 \cap \mathbb{Z}_2 \supseteq \mathbb{Z}_1 \cap \mathbb{Z}_2 \cap \mathbb{Z}_3 \supseteq \ldots \).
and it eventually stabilizes by Noetherian hypothesis. \( \Rightarrow \{Z_n \cap T\} \) eventually stabilizes.

(2). Let \( \mathcal{X} = \{ \text{closed subsets of } X \text{ which do NOT have finitely many irreducible components} \} \). If \( \mathcal{X} \) were not empty, by Noetherian hypothesis, it would contain a minimal element \( Z \). \( Z \) couldn't be irreducible since \( Z \in \mathcal{X} \Rightarrow Z = Z' \cup Z'' \), \( Z' \), \( Z'' \) closed and not equal to \( Z \Rightarrow Z' = \bigcup_{i=1}^{n} Z_i \) \( Z'' = \bigcup_{j=1}^{m} Z_j \) with \( Z_i, Z_j \) irreducible \( \Rightarrow Z = \bigcup_{i=1}^{n} Z_i \cup \bigcup_{j=1}^{m} Z_j \) . contradiction.

(3). If \( X = Z_1 \cup \ldots \cup Z_m \), then for each \( Z_i \), take \( Z_i \setminus (\bigcup_{j \neq i} Z_j) \)

Rmk: The argument in (2) is called "Noetherian induction".

Since we know that there is a 1-1 correspondence between minimal primes of a ring \( R \) and the irreducible components of \( \text{Spec} R \).

Cor. If \( R \) is a Noetherian, then \( R \) has only finitely many minimal primes. Thus \( \sqrt{(0)} = (\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r) \) , intersection over these minimal primes.

- Artin–Rees Lemma.
  
  Lemma: \( R \): Noetherian

(a). Any finite \( R \)-module is of finite presentation
(b). Any submodule of a finite \( R \)-module is finite.

Pf: (b) \( \Rightarrow \) (a): If \( M \) is finite \( \Rightarrow M \) is finitely presented: \( M = R^{\oplus n}/N \) for some \( n \) and \( N \subseteq R^{\oplus n} \). Now by (b), \( N \) is finite, i.e. \( N = R^{\oplus m}/K \) for some \( m \) and \( K \subseteq R^{\oplus m} \)

\( \Rightarrow R^{\oplus m} \rightarrow R^{\oplus n} \rightarrow M \rightarrow 0 \) is a finite presentation of \( M \) as an \( R \)-module.

Thus it suffices to prove (b).

Let \( N \subseteq M \) be a submodule. We will prove that \( N \) is finitely generated by induction on the number of generators of \( M \).

\( M \) is generated by 1 element: \( M = R/I \supseteq N \Rightarrow N \cong J/I \) for some \( I \subseteq J \subseteq R \).

Since \( R \) is Noetherian, \( J \) is finitely generated, then so is \( J/I \cong N \).
Induction step: Suppose $M$ is generated by $x_1, \ldots, x_n$. Let $M'$ be the submodule generated by $x_1, \ldots, x_{n-1} \Rightarrow M/M'$ is generated by 1 element, namely $x_n$.

$$0 \to M' \to M \to M/M' \to 0$$
$$0 \to N \cap M' \to N \to N/N \cap M' \to 0$$

By induction hypothesis, $N \cap M'$ and $N/N \cap M'$ are finitely generated. Chase a set of generators $\{x_1, \ldots, x_k\}$ for $N \cap M'$ and $\{y_1, \ldots, y_k\}$ for $N/N \cap M'$, where $y_j$'s are a lift of $\bar{y}_j$ in $N$. Then $N$ is generated by $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$.

Indeed, take any $z \in N$. $z$ in $N/N \cap M'$ can be written as $z = \sum y_j \bar{y}_j$. Thus $z = \sum y_j \bar{y}_j \mapsto 0$ in $N/N \cap M'$, and must be in $N \cap M'$. Therefore $z = \sum y_j \bar{y}_j = \sum x_i x_i \Rightarrow z = \sum \bar{y}_j x_i + \sum y_j \bar{y}_j$.

\[\square\]

Lemma (Artin-Rees Lemma)

$R$: Noetherian, $I \subseteq R$ is an ideal and $N \subseteq M$, finitely $R$-modules. Then:

$\exists \in \mathbb{N}$, s.t. $I^n M \cap N = I^{n+1} (I^n M \cap N)$ for all $n \geq c$.

Pf: Consider the ring $S = R \oplus I \oplus I^2 \oplus \cdots = \oplus_{n \geq 0} I^n$, where the multiplication is given by $I^n \times I^m \to I^{n+m}$, $(a, b) \mapsto ab$.

Suppose $I = (f_1, \ldots, f_n)$, then define $\psi: R[x_1, \ldots, x_n] \to S$ by $\psi(x_i) = f_i \in I$. Since $I^n = \text{ideal generated by products of the form } a_1 \cdots a_n, a_i \in I$

\[= \text{ideal generated by products of the form } f_1 \cdots f_n, 1 \leq j \leq n\]

Therefore, deg $m$ part of $R[x_1, \ldots, x_n]$ surjects onto $I^n$ via $\psi$, and thus $S$ is Noetherian. Similarly, consider the module $\tilde{M} = M \oplus IM \oplus I^2 M \oplus \cdots = \oplus_{n \geq 0} I^n M$ as an $S$-module $S \times \tilde{M} \to \tilde{M}$, $I^n \times I^m M \to I^{n+m} M$, multiplication given in the obvious fashion.

Claim: $\tilde{M}$ is a finite $S$-module. Namely, if $x_1, \ldots, x_n \in M$ generate $M$ as an $R$-module, then they also generate $\tilde{M}$ as an $S$-module.

Consider the $S$-submodule $\tilde{M}'$ of $\tilde{M}$ given by $\tilde{M}' = \oplus_{n \geq 0} I^n M \cap N \subseteq \tilde{M}$.

By the previous lemma, $\tilde{M}'$ is also finitely generated as an $S$-module, say by $\tilde{g}_j \in \tilde{M}'$. By taking their homogeneous components, we may assume that each $\tilde{g}_j$ is homogeneous in $I^n M \cap N$ for some $n_j$, $j = 1, \ldots, N$. 
Let $C \triangleq \max\{n_j \mid j=1,\ldots,n\}$. Any $\varpi \geq C$, and $\varnothing \in \mathcal{I}^n \cap \mathcal{N}$

We may write $\varnothing = \sum S_j \varpi_j$. Since the homogeneous components of $S_j$ other than those in $\mathcal{I}^{n_j} \cap \mathcal{N}$ will be canceled anyway, we may assume that $S_j \in \mathcal{I}^{n_j} \cap \mathcal{N}$. Moreover, since $C \geq n_j$, $\forall j$, $\mathcal{I}^{n_j} = \mathcal{I}^{n-C} \cap \mathcal{I}^{n_j}$,

$\Rightarrow \varnothing \in \sum \mathcal{I}^{n_j} (\mathcal{I}^n \cap \mathcal{N}) \subseteq \mathcal{I}^{n-C} (\mathcal{I}^n \cap \mathcal{N})$.

The other inclusion $\mathcal{I}^{n-C} (\mathcal{I}^n \cap \mathcal{N}) \subseteq \mathcal{I}^n \cap \mathcal{N}$ is obvious. \qed
§4. Jacobson Rings

In many cases rings will be Jacobson rings and the spectra will be Jacobson spaces i.e. if \( X_0 = \{ \text{closed points of } X \} \), \( X_0 \) is "very dense" in \( X \) i.e. \( \overline{Z} \) closed in \( X \) \( \Rightarrow \overline{Z} = \overline{Z \cap X_0} \).

- Thm (Hilbert Nullstellensatz)

Let \( k \) be a field.

(1). Any maximal ideal \( m \subseteq k[x_1, \ldots, x_n] \) has its residue field \( k(m) \) a finite field extension of \( k \).

(2). Any radical ideal \( I \subseteq k[x_1, \ldots, x_n] \) is the intersection of the maximal ideals containing \( I \).

Furthermore, the same holds in any finite type \( k \)-algebra in place of \( k[x_1, \ldots, x_n] \).

Pf: We will prove (1) by induction on \( n \).

\( n = 0 \) is clear.

Suppose \( m \subseteq k[x_1, \ldots, x_n] \) is maximal. Let \( \beta \) be the contraction of \( m \) in \( k[x_n] \).

Case I. \( \beta \neq 0 \Rightarrow \beta \) is maximal, generated by a monic irreducible \( p \in k[x_n] \) since \( k[x_n] \) is a P.I.D. Then \( k'[\subseteq k[x_n]/\beta] \) is a finite field extension of \( k \). And we have a surjection \( k'[\subseteq x_n, \ldots, x_{n-1}] \rightarrow k(m) \) given by \( \psi(x_i) = \overline{x_i} \) in \( k(m) \) and \( \psi(\overline{x_n}) = \overline{x_n} \) (\( [x_n] \) standing for the class of \( x_n \) in \( k' \)).

By induction hypothesis applied to \( k'[\subseteq x_n, \ldots, x_{n-1}] \), \( k'[\subseteq x_n] \) is a maximal ideal and \( k(m) \) is a finite field extension of \( k' \). Since \( k' \) is also finite over \( k \), we deduce that \( k(m) \) is finite over \( k \).

Case II. \( \beta = 0 \). Consider the ring extension \( k[x_n] = k[x_n]/(m) \rightarrow k[x_1, \ldots, x_n]/m = k(m) \)

\( \Rightarrow \text{Spec}(k(m)) \rightarrow \text{Spec}(k[x_n]) \). Since the ring map is finitely presented (\( m \) being finitely generated since \( k[x_1, \ldots, x_n] \) is Noetherian), by Chevalley's thm, the image of \( \phi \), i.e. the generic point of \( \text{Spec}(k[x_n]) \), is constructible, hence is non-empty and of the form \( \bigcup U_i \cap V_i^c \), \( U_i, V_i \) open and retrocompact (q.c.). If \( \{0\} \subseteq V_i^c \Rightarrow V_i^c \supseteq \{0\} = \text{Spec}(R) \Rightarrow \{\text{generic point}\} \) is non-empty open, thus would contain all but finitely many closed points, which is absurd. Thus \( \beta \) can't be 0.
The last claim of the thrn. is true since maximal ideals of \( k[x_1, \ldots, x_n]/J \) are in 1–1 correspondence with maximal ideals containing \( J \).

For (2), let \( I \subseteq k[x_1, \ldots, x_n] \) be a radical ideal, we have to show that \( I = \bigcap_{m \in I} m \) (= intersection of maximal ideals containing \( I \)).

Pick \( f \in I \), and we need to show that, \( \exists \) a maximal ideal \( m \supseteq I \), \( f \notin m \).

For this purpose, consider \( \text{Def} f \cap V(I) = \text{Spec}((R/I)_{f}) \). Since \( I \) is radical, \( f \) is not nilpotent mod \( I \Rightarrow \text{Def} f \cap V(I) \neq \emptyset \). Pick a closed point \( \overline{m} \) in it, which is a maximal ideal of \( (R/I)_{f} \), and consider its preimage in \( R \), denoted by \( m \). Then we have \( k \subseteq R/m \subseteq k(\overline{m}) \). By part (1), \( k(\overline{m}) \) is finite over \( k \). This implies that \( R/m \) is a field and \( m \) is a maximal ideal not containing \( f \).

Similar as above, the case for any finite type \( k \)-algebra is true since any radical ideal of \( k[x_1, \ldots, x_n]/J \) is a radical ideal of \( k[x_1, \ldots, x_n] \) containing \( J \). \( \square \)

Some Topology

Def: Let \( X \) be a topological space and \( X_0 \subseteq X \) be the set of closed points. We say that \( X \) is Jacobson if every closed subset \( Z \) of \( X \) is the closure of \( Z \cap X_0 \).

A Counter Example:

\[ \text{Spec } \mathbb{Z}_{(2)} = \{ \eta, s \} \] where \( \eta = (0) \) is the generic point and \( s = (2) \mathbb{Z}_{(2)} \) is the only closed point. The topology on \( \text{Spec } \mathbb{Z}_{(2)} \) is given by \( \{ \emptyset, \{s\}, \{\eta, s\} \} \) (as closed sets). Thus \( X_0 = \{s\} \) and \( \overline{X_0} = \{s\} \neq X \).

Remark: \( X \) Jacobson \( \Rightarrow X_0 \rightarrow X \) induces a bijection of closed subsets of \( X \) and closed subsets of \( X_0 \) (with the induced topology). The same thing holds for open subsets (just take the complement). Thus it's equally well to do sheaf theory on \( X \) or \( X_0 \), and the Krull dimensions of \( X \) and \( X_0 \) are the same.

Lemma:

The irreducible closed subsets of \( X \) satisfy \( Z = \overline{Z \cap X_0} \Rightarrow X \) Jacobson.
Pf: Take $Z$ a closed subset of $X$, then $Z = \bigcup Z_i$, union of irreducible components.
$\Rightarrow Z_i = \overline{Z_i \cap X_0} \Rightarrow Z \supseteq \overline{Z \cap X_0} = \overline{U (Z \cap X_0)} \supseteq \overline{U Z \cap X_0} = U Z_i = Z$. □

Lemma:

The property of $X$ being Jacobson is local, i.e. $X = \bigcup_{i \in I} U_i$, $U_i$ open, then $X$ being Jacobson $\iff$ each $U_i$ is Jacobson.

Pf: $\Rightarrow$ Any closed subset of $U$, $Z = \overline{Z \cap U}$. $X$ being Jacobson $\Rightarrow \overline{Z} = \overline{Z \cap X_0}$
Moreover, $\overline{Z} = (\overline{Z \cap U}^g) U \overline{Z \cap U} = \overline{(Z \cap U)^g} U Z$. Thus $\overline{Z} = \overline{Z}$

Lemma:

$X$ Jacobson. The following types of subsets $T$ are Jacobson and $T_0 = T \cap X_0$

1. open subsets
2. closed subsets
3. locally closed subsets
4. finite union of locally closed subsets. ( $\equiv$ constructible subsets )
Back to algebra:
Def: $R$ a ring. We say $R$ is Jacobson if every radical ideal $I$ is the intersection of maximal ideals containing it.

Cor. Any finite type algebra over a field is Jacobson. (by Nullstellensatz).

Lemma:
A ring $R$ is Jacobson $\iff$ $\text{Spec } R$ is a Jacobson space.

Pf: $\Leftarrow$ Take a radical ideal $I \subseteq R$. Let $J = \bigcap_{m \supseteq I} m$. Then $J \supseteq I \Rightarrow V(J) \subseteq V(I)$ and $V(J)$ is the closure of $V(I) \cap (\text{Spec } R)$ in $\text{Spec } R$, but this is equal to $V(I)$. Moreover, $J$ is a radical ideal since it's the intersection of radical ideals. $\Rightarrow V(J) = V(I)$ and $I = \sqrt{J} = \overline{J}$.

$\Rightarrow$ If $Z = V(I) \subseteq \text{Spec } R$ is a closed subset, we shall show that the set of closed points is dense in $Z$, or in other words, every $\text{Def}_f \cap V(I)$ contains a closed point of $Z$, if $\text{Def}_f \cap V(I) \neq \emptyset$. Here we may assume $I$ is radical and $f \in I$. Since $R$ is Jacobson, $I = \bigcap_{m \supseteq I} m \Rightarrow \exists m \supseteq I$ and $f \in m \Rightarrow m \in \text{Def}_f \cap V(I)$.

Lemma:
$\mathcal{Z}$ is Jacobson.

Pf: Every non-0 radical ideal is of the form $(n)$ where $m^2 \nmid n$, $\forall m, m^2 \neq 1$. Thus $n = \prod_{p_i}$, $p_i$ distinct $\Rightarrow (n) = \cap (p_i)$.
Moreover, $0 = \cap (p_i)$ over all prime ideals.

Lemma:
Let $R \longrightarrow S$ be a ring map, $m$ a maximal ideal in $R$, and let $g$ be an ideal lying over $m$. If $k(m) \subseteq k(g)$ is algebraic, then $g$ is maximal.

Pf: Consider the diagram on the right:

\[
\begin{align*}
S & \longrightarrow S/\overline{g} \hookrightarrow k(g) \\
\uparrow & \quad \uparrow \quad \uparrow \\
R & \longrightarrow R/m = k(m)
\end{align*}
\]

$\Rightarrow S/\overline{g}$ is a field.
$\Rightarrow g$ is maximal.
Prop: Let $R$ be a Jacobson ring and let $R \rightarrow S$ be of finite type, then:

1. $S$ is Jacobson.
2. $\text{Spec} \ S \rightarrow \text{Spec} \ R$ maps closed points to closed points.
3. If $m' \subseteq S$ is maximal lying over $m \subseteq R$, then the residue extension $k(m) \subseteq k(m')$ is finite.

Before giving the proof, we see that:

- Any finite type $\mathbb{Z}$-algebra is Jacobson.

- If $R$ is of finite type over $\mathbb{Z}$ and $m \subseteq R$ is a maximal ideal, then $\mathbb{Z} \cap m = (p)$ with $p$ prime and $R/p = \mathbb{Z}/p \rightarrow k(m)$ is a finite field extension.

Proof of prop.

Write $S = R[x_1, \ldots, x_n]/I$, then we see that it suffices as before to prove the prop. for two cases: (i) $S = R/I$; (ii) $S = R[x]$

Case (i) is easy, since $\text{Spec} R/I \subset \text{Spec} R$ is a closed immersion thus $\text{Spec} R/I$ is Jacobson as a topological space, thus $R/I$ is Jacobson. (2) and (3) follow.

Case (ii): By a top lemma, to show that $\text{Spec} R[x] \subset \text{Spec} R$ it suffices to show for irreducible subsets of $\text{Spec} R[x]$. Thus take a prime in $R[x]$, $\mathfrak{p} \subsetneq q \cap R$. Take $f \in R[x]$, $f \not\in q$, we need to show that $\text{Def}(f) \cap V(q)$ contains a closed point.

Consider the diagram on the right. We are reduced to consider the case where $\beta = 0$ and $R$ is a domain.

By case (i) $R$ is also Jacobson.

(I). If $\mathfrak{a} = (0)$. Write $f = ax_1 + \cdots + a_n$. Since $\mathfrak{a} \subseteq \mathfrak{q}$, not all $a_i$ are zero. Since $R$ is Jacobson, we can find a maximal ideal $m$ of $R$ such that $a_i \in m$ for some $i$.

Thus $f \not\in \mathfrak{m}$ in $k(m) \otimes R[x] = k(m)[x]$, and we can choose a maximal ideal not containing the image of $f$. Since $k(m)[x]$ is Jacobson by Hilbert Nullstellensatz, and thus the inverse image $m' \subseteq R[x]$ defines a closed point of $\text{Def}(f) \cap V(q)$ and $k(m')$ is finite over $k(m)$, by Nullstellensatz again.

(II). If $\mathfrak{a} \neq (0)$. Then $0 \rightarrow R \rightarrow \frac{R[x]}{\mathfrak{a}} \rightarrow 0 \rightarrow K \rightarrow K[x]/gK[x]$ where $K$ is the fraction field of $R$. Since $k[x]$ is a P.I.D., $k[x] \subseteq (g)$ for an irreducible polynomial.
\[ g \in K[x]. \] We may further assume \( g = b_0 x^e + \cdots + b_n, b_i \in R, b_0 \neq 0. \) Since \( R_{be} \) is also Jacobson, and \( R_{be} \twoheadrightarrow R, R[x] \twoheadrightarrow R_{be}[x] \) induces isomorphisms on residue fields, we are reduced to the case where \( g \) is monic. Then \( R[x]/g \cong R^{\otimes e} \subseteq K[x]/g \cong K^{\otimes e} \Rightarrow g = (g). \) By lemma ???, the image of \( \text{Def}(\bigvee Vg) \) in \( \text{Spec} R \) is \( \text{Def}(\bigvee Vg) \) for some \( \epsilon \in R. \) Take any maximal ideal in this image, say \( m \), and consider any prime \( m' \) lying over \( m. \) Since \( m' \supseteq g, \) the residue field of \( m' \) is finite over \( K(m) \) because \( m(m) \subseteq \frac{R[x]}{m} \subseteq K(m) \Rightarrow m' \) is maximal and \( K(m') \) is finite over \( K(m). \)

Thus we have shown that \( \text{Spec} R[x] \) is Jacobson, and it follows for \( R[x]. \)

Moreover, take \( a \) to be any maximal ideal of \( R[x]. \) The above argument shows that \( m' \) is mapped to \( m \) (and \( K(m') \) is finite over \( K(m) \)). \[ \square \]

Remarks:

In case \( R \) is Noetherian and Jacobson (e.g. of finite type over \( k \) or \( \mathbb{Z} \)) \[ R \twoheadrightarrow S \text{ finite type } \iff \text{Spec} S \xrightarrow{f} \text{Spec} R \] the induced map on the spectrum.

Consider the diagram:

\[
\begin{array}{ccc}
\{ \text{Constructable subsets of } \text{Spec} R \} & \longrightarrow & \{ \text{Constructable subsets of } (\text{Spec} R)_0 \} \\
\uparrow f & & \uparrow f_0 \\
E & \longrightarrow & E_0
\end{array}
\]

\[
\begin{array}{ccc}
\{ \text{Constructable subsets of } \text{Spec} S \} & \longrightarrow & \{ \text{Constructable subsets of } (\text{Spec} S)_0 \} \\
\uparrow f & & \uparrow f_0 \\
E & \longrightarrow & E_0
\end{array}
\]

where the horizontal maps are bijections. Note that \( f_0 \) is well-defined by the previous proposition, and the diagram commutes.
Def. If $\mathbb{A} = \mathbb{A}$, and $R, S$ are of finite type over $\mathbb{A}$.

$R \rightarrow S \rightarrow Y \cong \text{Spec} S \rightarrow \text{Spec} R \cong X$

$\downarrow$

Then $X_0, Y_0$ are the \textquoteleft $\mathbb{A}$-points\textquoteright\ of $X$ and $Y$, in the sense that:

$R \rightarrow R/m \cong \mathbb{A} \Rightarrow \text{we have a } \mathbb{A}\text{-algebra homomorphism } R \rightarrow \mathbb{A}$

$\Rightarrow \text{Spec } \mathbb{A} \rightarrow \text{Spec } \mathbb{A}$, the image of the closed point $m$.

So $X_0 \cong \text{Hom}_{\mathbb{A}}(R, \mathbb{A}) \leftarrow \text{Hom}_{\mathbb{A}}(S, \mathbb{A}) \cong Y_0$

Example:

$X = \text{Spec}(k[[t_{ij}]_{ij\in A}]) \rightarrow \text{Spec}(k[[t_{ij}]]/(T^2 - T)) \Rightarrow k = \bar{k}$. Here $(T^2 - T)$ stands for the ideal generated by the entries of $T^2 - T$. Then $X_0$ is in 1-1 correspondence with the set of idempotent matrices.

Define: $GL_n \rightarrow X: g \mapsto g \left( \begin{smallmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \end{smallmatrix} \right) g^{-1}$, where $GL_n = \text{Spec}(k[[g_{ij}]_{ij\in A}])$ and rank $\left( \begin{smallmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \end{smallmatrix} \right) = r$. Then $g_r$ corresponds to the ring map:

$k[t_{ij}]/(T^2 - T) \rightarrow k[[g_{ij}]]$

$t_{ij} \mapsto \text{the } (i,j)-\text{th entry of } g \left( \begin{smallmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \end{smallmatrix} \right) g^{-1}$

From Jordan canonical form we know that every idempotent matrix can be written in the form $g \text{diag}(I_r, 0_{n-r}) g^{-1}$. Thus the image of $g_r$ in $X_0$ consists of all idempotents of $M_n(k)$. Since $GL_n$ is irreducible, its image is also irreducible. When $r$ ranges from 0 to $n$, the image ranges over all $X$. Moreover we see that we can cook up \textquoteleft functions\ taking different values on different components $g_r(GL_n)$, $r = 0, \ldots, n$, namely $f \in k[t_{ij}]/(T^2 - T)$, $f(t_{ij}) = \sum t_{ii}$ (Assume $\text{char } k = 0$), and $f$ takes the value \text{r} on $g_r(GL_n) \Rightarrow f - r \in \bigcap M = \beta_r \cong \text{the minimal prime corresponding to the component } g_r(GL_n)$. Hence we can see that $X_0$ is disconnected.

(In char $k > 0$, this $f$ need not separate all components, for instance, char $k = 3$

$\text{tr} \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ \end{smallmatrix} \right) = 3 \neq 0 \text{ and } \text{tr} \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ \end{smallmatrix} \right) = 0$. However, we may cook up more sophisticated versions of \textquoteleft tr\textquoteright, namely, we may take the trace of $\Lambda^3 \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ \end{smallmatrix} \right)$ or $\Lambda^3 k^3 = k$, which is 1 on $g_3(GL_3)$ but 0 on any other components $g_i(GL_3), i = 0, 1, 2$.)
\section{Artinian Rings}

\begin{itemize}
\item Nakayama's Lemma
\end{itemize}

\begin{enumerate}
\item If \( M \) is finite, \( IM = M \), then \( \exists x \in I \text{ s.t. } (1 + x)M = 0 \).
\item If \( M \) is finite, \( IM = M \) and \( I \subseteq \text{Rad}(R) \) (Rad = \( \bigcap \) all maximal ideals of \( R \)), then \( M = 0 \).
\end{enumerate}

\textbf{Pf.} \ ((a)) Pick generators \( m_1, \ldots, m_n \) of \( M \). Then \( M = IM \Rightarrow m_i = \sum f_{ij} m_j, \quad f_{ij} \in I \Rightarrow A = (\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, (\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix})^t) \text{ annihilates the column vector } (m_1, \ldots, m_n)^t.
\Rightarrow 0 = A^*A (\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}) = (\begin{pmatrix} \det A \\ \vdots \\ \det A \end{pmatrix}) (\begin{pmatrix} \det A \\ \vdots \\ \det A \end{pmatrix}) \Rightarrow \det A \text{ annihilates } M.
\text{But } \det A = \det (I + (f_{ij})) = 1 + x, \quad x \in I.
\text{ (b) It suffices to show that } 1 + x \text{ is invertible. Suppose not, then } 1 + x \in M \text{ for some maximal ideal, but } x \in \text{Rad}(R) \subseteq M \Rightarrow 1 \in M, \text{ which is absurd.}\]

\textbf{Rmk.} Typically, Nakayama's lemma is applied where \( R \) is local and \( I \) is the maximal ideal of \( R \). Then if \( M \) is finite, any generator of the finite dimensional \( R/m \)-vector space \( M/mM \) lifts to generators of \( M \).

\textbf{Pf.} Consider the submodule of \( M \) generated by the lifts, say, \( m_1, \ldots, m_n \), then \( \frac{M}{m + \langle m_m \rangle} = 0 \Rightarrow m \frac{M}{m + \langle m_m \rangle} = \frac{mM + \langle m_1, \ldots, m_n \rangle}{m + \langle m_1, \ldots, m_n \rangle} = 0. \text{ People tend to make mistakes by forgetting to check } M \text{ finitely generated.}

As an application of Nakayama's lemma, we prove:

\begin{itemize}
\item Lemma:
\item Let \( R \) be a Noetherian local ring, \( M \) finite module over \( R \). \( I \subseteq R \) is a proper ideal, then \( \bigcap_{m \in I} M = 0 \). In particular, if we take \( R = M = k[x_1, \ldots, x_n], \quad I = m = (x_1, \ldots, x_n)R \text{ the maximal ideal, then } \bigcap_{m \in I} M = 0 \).
\item Pf. Take \( N = \bigcap_{m \in I} M \). By assumption, \( N \) is finitely generated.

By Artin-Rees lemma, \( \exists c \in I \text{ s.t. } I^c M \cap N = I^c (IM \cap N) \). But in our case, \( I^k M \cap N = N, \forall k. \text{ Hence if } n = c + 1, \quad N = IN. \text{ Nakayama } \Rightarrow N = 0. \)
\end{itemize}
• Length of an \( R \)-module.

**Def:** \( R \) is a ring and \( M \) an \( R \)-module. The length of \( M \) as an \( R \)-module is defined as:
\[
\text{length}_R M = \text{Sup}\{n \mid \exists \text{ filtration: } M_0 \supset M_1 \supset \cdots \supset M_n = M\}
\]

**Example:**
\( \mathbb{R}[x] \), \( M_1 = R \), then \( \text{length}_R (M_1) = +\infty \), since in this case, \( \forall n \in \mathbb{N} \) we may always find filtrations of length \( n \): e.g. \( (x^n) \supseteq (x^{n-1}) \supseteq \cdots \supseteq (x) \supseteq M_1 \).

On the other hand, \( M_2 = \mathbb{R}[x]/(x^5) \) is a module of length 5: Since the only maximal filtration is given by \( \mathbb{R}[x]/(x^5) \supseteq (x)/(x^5) \supseteq \cdots \supseteq (x^4)/(x^5) \supseteq 0 \), any two subsequent quotients \( \simeq \mathbb{R} \), the simple finite \( \mathbb{R}[x] \)-module. \( \Rightarrow \text{length}_R (M_2) = 5 \).

**Fact:** Equivalence of category
\[
\{ \text{Finite length } \mathbb{R}[x]-\text{modules} \} \leftrightarrow \{ \text{Pairs } (V, T) \text{ where } V \text{ is a finite dimensional vector space with a linear operator acting on it} \}
\]

**Lemma:**
\[
0 \to M' \to M \to M'' \to 0 : \text{a short exact sequence of } R\text{-modules. Then}
\]
\[
\text{length}_R (M) = \text{length}_R (M') + \text{length}_R (M'')
\]

**Pf:** Given filtrations for \( M', M'' \) respectively of length \( n', n'' \) (may be \( \infty \)) say:
\( M' = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \) \( \supseteq 0 \)
\( M'' = M_1'' \supseteq M_2'' \supseteq \cdots \supseteq M_n'' \) \( \supseteq 0 \).
Then the filtration on \( M \) given by:
\( M = M_1 \supseteq M_1' \supseteq M_1'' \supseteq \cdots \supseteq M_n \) \( \supseteq 0 \) of length \( \geq n' + n'' \).

Conversely, suppose \( M_0 \supseteq \cdots \supseteq M_n = M \) is a filtration of \( M \) of length \( n \). Consider \( M_i = M_i \cap M' \), \( M_{ii} = \varphi_i (M_i) \).
Let \( n' = \# \{ \text{strict inclusions in } M_i \} \)
\( n'' = \# \{ \text{strict inclusions in } M_{ii} \} \).

**Claim:** If \( M_i = M_i' \), \( M_{ii} = M_{ii}' \), then \( M_i = M_{ii} \).

Indeed, we have by definition the exact sequences:
\[
0 \to M_i \to M_i' \to M_{ii}' \to 0
\]
\[
M_i \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
0 \to M_{ii} \to M_{ii} \to M_{ii}' \to 0
\]

Now if \( \ell', \ell'' \) are identities, by five lemma, so is \( \ell \).

Hence in our setting, either \( M_i \neq M_i' \) or \( M_{ii} \neq M_{ii}' \) \( \Rightarrow n' + n'' \geq \text{length}_R M \). \( \square \)
Lemma:
Suppose $R$ is a Noetherian local ring with maximal ideal $m$. $M$ a finite $R$ module.
If $m^nM \neq 0$, $\forall n$, then $\text{length}_R(M) (\equiv \ell_R(M) \text{ for short}) = +\infty$.

Pf: By Nakayama's lemma, all the inclusions in the filtration: $M \supseteq mM \supseteq m^2M \supseteq \ldots$ are strict. (otherwise $m^kM = m^{k+1}M \Rightarrow m:mm^kM = m^{k+1}M = m^kM \Rightarrow m^kM = 0$).

On the other hand, we have:

Lemma:
$R$: local Noetherian, $m \subseteq R$ its maximal ideal. $M$ an $R$-module and $mM = 0$.
Then $\ell_R(M) = \dim_{\mathbb{F}/m}(M)$.

Pf: Note that $M$ is a well-defined $R/m$-vector space. Any submodule is just a subspace.

Lemma:
$R$: Noetherian local ring, $M$: a finite $R$ module with $m^nM = 0$ for some $n$.
Then $\ell_R(M) < \infty$.

Pf: Consider the filtration, $M \supseteq mM \supseteq \ldots \supseteq m^nM = 0$. If each step is of finite length, then $\ell_R(M) = \sum_{i=0}^{n} \ell_R(m^iM/m^{i+1}M) < \infty$, by additivity of length.
But each $m^iM/m^{i+1}M$ is finite and killed by $m$, and thus is of finite length.

"Mathematics is a sequence of tautology..." — de Jong

Artinian Rings
Def: A ring is called Artinian if the descending chain condition holds for ideals.
($\Rightarrow$ a.c.c. for closed subsets of $\text{Spec}R$)

Examples:
1. Finite rings (e.g. $\mathbb{Z}/n$) are Artinian.
2. Given a field $k$, all finite dimensional $k$-algebras are Artinian. In fact, if $R$ is of finite type over $k$, then $R$ is Artinian $\iff \dim_k R < \infty$. 
Lemma:

If \( R \) is Artinian \( \Rightarrow \text{Rad}(R) \) is nilpotent.

**Pf:** Set \( I = \text{Rad}R \), and look at \( I \supseteq I^2 \supseteq \cdots \)

To get a contradiction, assume that \( J \neq R \). Pick \( J \neq J' \), \( J' \) a minimal element among ideals strictly containing \( J \). Pick \( x \in J' \setminus J \), then by minimality, \( J' = (x) + J \).

If \( J' \neq J \) is finite. Nakayama \( \Rightarrow IJ'/J \subseteq J' \Rightarrow IJ' \subseteq J \) by minimality again. \( \Rightarrow J'I^n \subseteq J^n = 0 \), but \( J'I^n = J'I^{n+1} = J'I^n \Rightarrow J' \subseteq J \), contradiction.

Lemma:

\( R \) a ring, \( I \subseteq R \) a nilpotent ideal. Then idempotents of \( R \) \( \iff \) idempotents of \( R/I \).

**Pf:** (Geometric):

\[ \begin{array}{ccc}
\{ \text{Idempotents of } R \} & \xrightarrow{\sim} & \{ \text{Idempotents of } R/I \} \\
\downarrow & & \downarrow \\
\{ \text{Connected components of } \text{Spec}R \} & \xleftarrow{\sim} & \{ \text{Connected components of } \text{Spec}R/I \} 
\end{array} \]

(Algebraic): this method (Newton) also works for non-commutative rings.

For any idempotent \( \bar{e} \) of \( R/I \), we need to find an idempotent \( e \) of \( R \) mapping to \( \bar{e} \).
First take any preimage of $\bar{e}$ in $R$, say $e_i$, then
\[ e_i^2 - e_i = x \in I \]
Choosing different lifts $e_2 = e_i + y$ gives
\[ e_2^2 - e_2 = e_i^2 + 2e_iy + y^2 - e_i - y \]
\[ = (2e_i - 1)y + y^2 + x \equiv 0 \mod I^2 \]
Indeed the equation can be solved since $2e_i - 1$ is a unit: $(2e_i - 1)^2 = 4(e_i^2 - e_i) + 1 = 1 + x$ and $x$ is nilpotent. Thus we may just take $y = \frac{x}{1-2e_i} \in I$, and the difference of $e_i^2 - e_2$ is in $I^2$. Inductively this can be done mod $I^n$ for all $n$, but since $I$ is nilpotent, $I^n = 0$ for some $n$, and $e_i^2 - e_i = 0$. \qed

**Upshot:** An Artinian ring is the product of its localizations at maximal ideals and the only primes are the maximals.
($R = \mathbb{Q} \times \mathbb{Q}$, localizations give copies of $\mathbb{Q}$, as well as taking quotient $\mathbb{Q}$.)

**Lemma:**
Any ring with finitely many maximal ideals and whose radical is nilpotent (locally nilpotent) is the product of its localizations at maximal ideals. Moreover, any prime is maximal.

**Pf:** Let $m_1, \ldots, m_n$ be the maximal ideals of $R$. $\text{Rad}(R) = \bigcap m_i$. Nilpotency of $\text{Rad}(R) \Rightarrow \text{Rad}(R) \subseteq \bigcap \beta \ni \beta$, for any prime $\beta$ of $R$. $\Rightarrow \beta \ni \bigcap m_i \Rightarrow \beta \ni m_i$ for some $i$. $\Rightarrow \beta = m_i$, thus maximal.

By Chinese remainder thm, we have $R/\text{Rad}(R) \cong R/m_1 \times \cdots \times R/m_n$. By the previous lemma, we can find idempotents $\bar{e}_1, \ldots, \bar{e}_n$ of $R/\text{Rad}(R)$ corresponding to $(0, \ldots, 1, \ldots, 0)$ in $R/m_1 \times \cdots \times R/m_n$ ($\bar{e}_i \bar{e}_j = \delta_{ij} \bar{e}_i$), whose lifts in $R$ give rise to the ring decomposition $R \cong R_{e_1} \times \cdots \times R_{e_n}$, which is a decomposition of $R$ into local rings. ($e_i \in m_i$ but $e_i \not\in m_j$, $\forall j \neq i$; $R_{e_i} \cong R_{e_i}$ which is local with maximal ideal $m_i$; $\text{Spec}(R_{e_i}) = \{m_i\}$). \qed

**Lemma:**
A ring $R$ is Artinian iff it is a module of finite length over itself. If so, $R$
is Artinian and Noetherian, and is equal to products of localizations at maximal ideals.

Pf: \( l_R(R) < \infty \iff \) both a.c.c. and d.c.c. hold for \( R \)
\[ \iff R \text{ is Noetherian and Artinian.} \]

It suffices to show that, \( R \) Artinian \( \Rightarrow l_R(R) < +\infty \).

Now \( R \) Artinian \( \Rightarrow R = R_1 \times \cdots \times R_m \), with \( R_i \) local Artinian. Moreover, by \( 0 \rightarrow R_i \rightarrow R_1 \times \cdots \times R_i \rightarrow R_1 \times \cdots \times R_{i-1} \rightarrow 0 \) and additivity of the length function it suffices to prove for the Artinian local case.

Now, \( R \geq m \geq \cdots \geq m^n \geq m^{n+1} = 0 \). \( \Rightarrow l_R(R) = \sum_{i=0}^n l_R(m_i/m_{i+1}) = \sum_{i=0}^n \dim_{R/m} (m_i/m_{i+1}). \)

Moreover, each \( m_i \) as \( R \)-module also satisfies d.c.c. \( \Rightarrow \dim_{R/m} m_i/m_{i+1} < \infty. \) \( \square \)
§6. $K$-Groups

$R$: a ring. We define two functors: $K_0(R), K_0(R) : \text{Mod}_R \to \text{Ab}.$

Def: $(K_0(R))$ For every finitely generated $R$-module $M$, $\exists$ an element $[M] \in K_0(R)$ and for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite $R$-modules $[M] = [M'] + [M'']$ in $K_0(R)$. Furthermore, it’s the free object with these two properties.

Def: $(K_0(R))$ For every finite projective $R$-module $M$, $\exists$ an element $[M] \in K_0(R)$, and for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite proj. $R$-modules $[M] = [M'] + [M'']$ in $K_0(R)$. Furthermore, it’s the free object with these two properties.

(About freeness: What you see is what you get! — de Jong)

Lemma:

Let $R$ be an Artinian ring. Then $\text{length}(-)$ induces an abelian group homomorphism $K_0(R) \to \mathbb{Z}$, $[M] \mapsto \text{length}_R(M)$.

Pf: Any $M \cong R^n/N$ is of finite length, since $R$ is. Furthermore, $\text{length}_R$ is additive. $\square$

Note that there is a canonical homomorphism: $K_0(R) \to K_0(R)$

Examples:

(i). $R = k$ a field, then $K_0(R) \cong K_0(R) \cong \mathbb{Z}.$

(ii). Let $k$ be a field, $K_0(k[x]) \cong K_0(k[x]) \cong \mathbb{Z}.$

Indeed, by the structure thm of finitely generated modules over a principal ideal domain, any $M$ finitely generated, then $M \cong R^n \oplus \bigoplus \frac{R}{(f_i)}$, if irreducible.

Hence $M$ projective $\iff$ $M$ free. $K_0(R) \cong \mathbb{Z}$ trivially.

Furthermore, we can show that: if $N$ torsion ($N \cong R/(f^n)$), then $[N] = 0$ in $K_0(R)$. Indeed, $0 \to R \xrightarrow{f} R \to N \to 0 \Rightarrow [R] = [R] + [N] \Rightarrow [N] = 0.$

Now rank : $K_0(R)$ is well-defined since rank is additive.

Now if $\text{rank}(\Sigma n_i[M_i]) = 0 \Rightarrow \Sigma n_i \text{ rank } M_i = 0$. Group together those $n_i$ which are positive or negative. We obtain: $\Sigma n_i \text{ rank } M_i = k \Rightarrow \Sigma n_i \text{ rank } M_i > 0$.
\[ 0 \rightarrow R^k \oplus R^{k'} \rightarrow R^k \oplus R^{k'} \rightarrow \oplus M_i^{n_i} \rightarrow 0 \quad (n_i > 0 \text{ or } n_i < 0) \]

\[ \Rightarrow \sum_{n_i > 0} n_i [M_i] - \sum_{n_i < 0} n_i [M_i] = \sum_{n_i > 0} [R^{k+k'*}] - \sum_{n_i > 0} [R^{k+k'*}] + \sum_{n_i < 0} [R^{k+k'*}] - \sum_{n_i < 0} [R^{k+k'*}] = 0. \]

(3) Let \( k \) be a field, \( R = \{ f \in k[x] \mid f(a) = f(1) \} \). In this case \( K_0(R) \cong k^* \oplus \mathbb{Z} \)

\( K_0(R) \cong \mathbb{Z} \).

\( R \cong k[A, B]/(A^3 - B^2 + AB) \)

\[ \frac{k[A, U]}{(A - u^2 + u)} \rightarrow \frac{k[A, B]}{(A - u^2 + u)} \rightarrow \oplus e_i \]

\[ \cong k[U] \oplus \cdots \oplus k[U] \]

\[ \frac{k[B, S]}{(-1 + S + B)} \rightarrow \oplus e_i' \rightarrow k[S] \oplus \cdots \oplus k[S] \]

\[ B^3 S^3 - B^3 + B^3 S \]

\[ \lambda + \frac{1}{S} f'(s) \]

\[ S(1 + BS^3) = 1 \]
Fact: $K_0(R \times R_2) \cong K_0(R) \times K_0(R_2)$; $K_0(R \times R_2) \cong K_0(R) \times K_0(R_2)$.

Lemma:
Suppose $R$ is local, $K_0(R)^{\text{rank}} \cong \mathbb{Z}$.

Pf: First of all we can show that if $R$ is local, then $M$ projective $\Rightarrow$ $M$ free.
Indeed, let $m_1, \ldots, m_n$ be a minimal set of generators $\Rightarrow 0 \rightarrow \ker \phi \rightarrow R^n \rightarrow M \rightarrow 0$ splits since $M$ projective. $\Rightarrow R^n = M \oplus \ker \phi \Rightarrow \exists \phi: K^n = M/mM \oplus \ker \phi/m\ker \phi$.

By Nakayama's lemma $M/mM \cong K^n \Rightarrow \ker \phi/m\ker \phi = 0 \Rightarrow \ker \phi = 0$, by Nakayama again. (note that ker $\phi$ is necessarily finitely generated, since $R^n \rightarrow \ker \phi$ is also surjective).

Thus $\text{rank}: K_0(R) \rightarrow \mathbb{Z}$ is well-defined, since $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ s.e.s of projective modules $\Rightarrow M' \cong R^n$, $M'' \cong R^n$, $M \cong R^{n+m}$ for some $n,m \in \mathbb{Z}$. \text{rank}(\Delta(G[R^n])) $= 0$ $\iff$ $0 \rightarrow \bigoplus_{i=0}^{n} R^{m_i} \rightarrow \bigoplus_{i=0}^{m} R^{a_i} \rightarrow 0$ exact. \hfill □

Lemma:
Suppose $R$ is Artinian local, then we have a commutative diagram:

\[
\begin{array}{ccc}
K_0(R) & \xrightarrow{\text{rank}} & K_0'(R) \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{Z} & \xrightarrow{\text{length}_R} & \mathbb{Z}
\end{array}
\]

Pf: Since the maps are well-defined, it suffices to check for generators of $K_0(R)$, namely free modules over $R$, which again reduces to check for $R$ itself: $\text{rank}_R(R) \cdot \text{length}_R(R) = \text{length}_R(R)$. \hfill □
§7. Graded Rings and Modules.

Def. (Graded Ring) \( S = \bigoplus_{d \geq 0} S_d \). \( S_d \cdot S_e \subseteq S_{d+e} \). \( S_+ = \bigoplus_{d \geq 1} S_d \) is an ideal of \( S \), called the irrelevant ideal.

Def. (Graded Module) \( M = \bigoplus_{d \geq 0} M_d \). \( S_d \cdot M_e \subseteq M_{d+e} \).

Lemma.

\( S \) is Noetherian \( \iff S_0 \) is Noetherian and \( S_+ \) is finitely generated as an \( S \)-module. Furthermore, homogeneous elements \( f_1, \ldots, f_n \in S_+ \) generate \( S \) as an \( S_0 \) algebra iff they generate \( S_+ \) as an ideal.

Lemma.

If \( M \) is a finitely generated graded \( S \)-module and \( S \) is noetherian, then each \( M_d \) is a finite \( S_0 \)-module.

Def. (Numerical Polynomial) \( A \) an abelian group. A function \( f: \mathbb{Z} \rightarrow A \) defined for all but finitely many integers is called a numerical polynomial if \( \exists r \geq 0 \), \( a_0, \ldots, a_r \in A \) s.t. \( f(n) = \sum_{i=0}^{r} \binom{n}{i} a_i \)

Examples

(a). \( S = k[x_1, \ldots, x_n] \), \( \deg x_i = 1 \), \( \dim_k k[x_1, \ldots, x_n] = \binom{n+d-1}{d-1} = f(d) \) is a numerical polynomial.

(b). \( S = k[x] \), \( \deg x = 2 \), \( \dim_k k[x] = 1 \) all odd can't be numerical.

Lemma.

Suppose \( f: n \mapsto f(n) \in A \) defined for \( n \gg 0 \) satisfies \( n \mapsto f(n) - f(n-1) \) is numerical, then \( f \) is numerical.

Pf.: Write \( f(n) - f(n-1) = \sum_{i=0}^{\infty} \binom{n}{i} a_i \). Then \( g(n) = f(n) - \sum_{i=0}^{\infty} \binom{n+i}{i} a_i \) satisfies \( g(n) - g(n-1) = f(n) - f(n-1) - \sum_{i=0}^{\infty} \left( \binom{n+i}{i} - \binom{n+i}{i+1} \right) a_i \)

\[ = f(n) - f(n-1) - \sum_{i=0}^{\infty} \binom{n}{i} a_i \]

\[ = 0 \quad \text{for } n \gg 0 \]
\[ g(n) = a_n \in A \text{ is a constant} \]
\[ f(n) = a_1 + \sum_{i=0}^{n} \frac{n+1}{i+1} a_i \quad \text{for } n \gg 0. \]

Now, given \( S \): Noetherian graded ring, \( M \): finite, graded \( S \)-module over \( S \). Define \( \Phi_M : d \mapsto [M_d] \in K_0(S) \)

Thm. Assume \( S^+ \) generated by elements in degree 1. Then the above function \( \ln(1) \to K_0(S) \) is a numerical polynomial.

Pf: Let \( M \) be generated over \( m_1, \ldots, m_s \), homogeneous elements of degree \( d_1, \ldots, d_s \). Now we prove by induction on the number of generators \( r \) of \( S^+ \) over \( S_0 \).

\( r = 0 \). \( S = S_0 \). Then \( M_d = 0 \) for \( d > \max(d_1, \ldots, d_s) \).

\( r > 0 \), let \( x \in S_1 \) be one of a minimal set of generators, so that we can apply the induction hypothesis to \( S/\langle x \rangle \approx S_0 \oplus \frac{S_1}{\langle x \rangle} \oplus \frac{S_2}{\langle x \rangle} \oplus \ldots \)

Case I: \( x \) is nilpotent on \( M \). This can be proved by induction on \( e \):
\[ x^e M = 0 \quad (M \text{ is killed by } x^e \text{ by Noetherian hypothesis}) \]
\[ e = 1 \]. \( M \) is a module over \( S/\langle x \rangle \). The above induction hypothesis applies.

Suppose it's true for \( < e \). For \( e \), we can find a short exact sequence of graded modules \( 0 \to M' \to M \to M'' \to 0 \) s.t. \( M', M'' \) are killed by \( x^{e'}, x^{e''} \), \( e', e'' < e \), then the induction hypothesis of this case applies. Such a short exact sequence can be taken, for instance, to be \( M' = \langle xM \rangle, M'' = M/\langle xM \rangle \). Then for each degree, we have \( 0 \to M'_d \to M_d \to M''_d \to 0 \) \( \forall d \in \mathbb{Z} \)

\[ \Phi_{M}(d) = \Phi_{M'}(d) + \Phi_{M''}(d) \]

\[ \Phi_M \] is a numerical polynomial since \( \Phi_{M'} \) and \( \Phi_{M''} \) are, by induction.

Case II. \( x \) is not nilpotent on \( M \). Let \( M' = \{ m \in M | x^t m = 0 \text{ for some } t \in \mathbb{N} \} \). Then \( M' \) is a finitely generated submodule of \( M \) by Noetherian hypothesis, we may then form \( 0 \to M' \to M \to M'' \to 0 \), where \( M'' \cong M/M' \), and \( x \) acts on \( M'' \) as a non-zero divisor. Since \( \Phi_M = \Phi_{M'} + \Phi_{M''} \), it reduces to prove for \( M'' \) on which \( x \) is a non-zero divisor.
Case III. \( x \) is a non-zero divisor on \( M \). Set \( \overline{M} = M/xM \). Then, \( \forall d \in \mathbb{N}, 0 \rightarrow M_{d-1} \xrightarrow{x} M_d \rightarrow \overline{M}_d \rightarrow 0 \) is short exact and \( \overline{M} \) is an \( S/xS \) module so that the induction hypothesis may be applied and \( \Phi_{\overline{M}} \) is a numerical polynomial. \( \Rightarrow \Phi_{\overline{M}}(d) - \Phi_{\overline{M}}(d-1) = \Phi_{\overline{M}}(d) \). Then \( \Phi_{\overline{M}}(d) \) is a numerical polynomial by our previous lemma. \( \square \)

Remark: Morphisms of graded modules \( \varphi : M \rightarrow N \) is an \( S \)-map and \( \varphi(M_d) \subseteq N_d \). Thus multiplication by homogeneous elements of \( S \) are not necessarily morphisms of graded modules. Yet the quotient module \( M/xM \) is still graded and \( M \rightarrow M/xM \) is a graded morphism.

Examples

(1). \( R = M = k[x_0, \ldots, x_n] \rightarrow R_0 = k, K_0(R_0) \twoheadrightarrow \mathbb{Z}, \dim_k M_d = \binom{d+n}{d} \)

(2). \( R = M = R/(f), f: \text{homogeneous of deg } d, \dim_k M_m = \binom{m+n}{m} - \binom{m-d+n}{m-d} \).

* Application: Noetherian Local Rings

\( R \): Noetherian local, with maximal ideal \( m \), residue field \( k = R/m \).

\( M \): a finite \( R \)-module.

Def: (Hilbert function of \( M \) over \( R \)) \( \varphi_M : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto \text{length}_R(M^n/M^mM) \)

(\( = \dim_k (M^n/M^mM) \), by Nakayama's lemma).

Note that \( \text{length}_R(M/m^nM) = \sum_{i=0}^{\dim M} \varphi_M(i) \).

Def: (a variant) Let \( I \subset R \) be an ideal of definition, i.e. \( \sqrt{I} = m \) (\( \iff m^n \subset I \) for some \( n \in \mathbb{N} \), in particular, we see that \( M/m^nM \rightarrow M/I^*M \), and thus \( M/I^*M \) has finite length). \( \varphi_{M,I}(n) \triangleq \text{length}_R(I^nM/I^{n+1}M) \)

Prop: \( \varphi_{M,I} \) is a numerical polynomial.

Pf: Consider the graded ring \( S = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \), and graded module \( N = M/IM \oplus IM/I^2M \oplus I^2M/I^3M \). Then \( S \) is a graded ring generated by degree
elements and $M$ is a finite $S$-module, and thus the theorem applies. Here we identify $K(S_0) = K(R/I) \cong \mathbb{Z}$ by taking length, since $R/I$ is Artinian. □

Lemma:
Suppose $0 \to M' \to M \to M'' \to 0$ is a s.e.s. of finite $R$-modules. Then there exists a submodule $N \subseteq M'$ of finite colength (i.e. $M'/N$ of finite length) and an integer $c \in \mathbb{N}$ s.t. $\varphi_{I,M}(n) = \varphi_{I,M'}(n) + \varphi_{I,N}(n)$, $n \gg 0$.

Proof: We have a surjective map $M/I^nM \xrightarrow{\varphi} M''/I^nM'' \to 0$, and $\ker \varphi = \frac{M'}{I^nM'}$.

$\Rightarrow 0 \to \frac{M'}{I^nM'} \to \frac{M}{I^nM} \to \frac{M''}{I^nM''} \to 0$ is a s.e.s.

Moreover, by Artin-Rees lemma $I^cM \cap M' = I^{n-c} (I^cM \cap M')$ for all $n \geq c$. Let $N = I^cM \cap M'$, then $N$ is of finite colength ($\frac{M'}{N} \cong \frac{M'}{I^cM} \cong \frac{M}{I^nM}$).

$\Rightarrow 0 \to \frac{M'}{I^nM} \to \frac{M}{I^nM} \to \frac{M''}{I^nM''} \to 0$ s.e for $n \geq c$.

Take length, and we obtain: $l_M(\frac{M}{I^nM}) = l_M(\frac{M'}{I^nM}) + l_N(\frac{M''}{I^nM''}) = l_M(\frac{M}{I^nM}) + l_N(\frac{M'}{I^nM'}) + l_N(\frac{M''}{I^nM''})$

$\Rightarrow \sum_{i=0}^{n} \varphi_{I,M'}(i) = \sum_{i=0}^{n} \varphi_{I,N}(i) + \sum_{i=0}^{n} \varphi_{I,M''}(i)$ for $n \geq c$.

Taking differences of neighboring $n, (n+1)$th equation

$\Rightarrow \varphi_{I,M}(n) = \varphi_{I,N}(n) + \varphi_{I,M''}(n)$. □

Lemma:
If $M' \subseteq M$ is of finite colength, then $\exists C_1, C_2 \in \mathbb{N}$ s.t.: $\varphi_{I,M'}(n) \geq \varphi_{I,M'}(n-C_1) - C_2$, and $\varphi_{I,M}(n) \geq \varphi_{I,M}(n) - C_2$.

Proof: Consider the s.e.s. $0 \to \frac{M'}{I^nM'} \to \frac{M}{I^nM} \to \frac{M''}{I^nM''} \to 0$. $M' \subseteq M$ of finite colength

$\Rightarrow IM \subseteq M'$ for some $C_1 \Rightarrow \text{For } n \geq C_1 \Rightarrow 0 \to \frac{M'}{I^nM'} \to \frac{M}{I^nM} \to \frac{M''}{I^nM''} \to 0$ is exact.

But $I^nM = I^{n-C_1} I^nM' \subseteq I^nM' \Rightarrow l_M(\frac{M'}{I^nM'}) \geq l_M(\frac{M}{I^nM})$. Hence

$l_M(\frac{M}{I^nM}) = l_M(\frac{M'}{I^nM'}) + l_M(\frac{M''}{I^nM''}) \geq l_M(\frac{M'}{I^nM'}) + l_M(\frac{M''}{I^nM'})$.

$\Rightarrow \varphi_{I,N}(n) \geq \varphi_{I,M''}(n-C_1)$ for $n \gg 0$.

Moreover $\frac{M'}{I^nM'} \to \frac{M}{I^nM} \Rightarrow l_M(\frac{M}{I^nM}) \geq l_M(\frac{M'}{I^nM'}) = l_M(\frac{M}{I^nM}) - l_M(\frac{M}{I^nM})$ by the s.e.s above.

$\Rightarrow \varphi_{I,M}(n) \geq \varphi_{I,M}(n) - C_2$ for $n \gg 0$.

Now take $C_2$ large, then $\varphi_{I,M}(n) \geq \varphi_{I,M'}(n-C_1) - C_2$. $\varphi_{I,M}(n) \geq \varphi_{I,M}(n) - C_2$ for all $n \in \mathbb{N}$.
Lemma:
Suppose I, I' are ideals of definition, and M a finite R-module. Then there exists an $a \in \mathbb{N}$ s.t. $\sum_{i=0}^{\infty} \phi_{I,M}(i) \leq \sum_{i=0}^{an} \phi_{I,M}(i)$.

Pf: Indeed, since $I^{a+1} \subseteq I$ for some $a \in \mathbb{N}$.

$\Rightarrow \sum_{i=0}^{\infty} \phi_{I,M}(i) = \ell_R(M/I^{i+1}M) \leq \ell_R(M/I^{(a-1)an+1}M) \leq \ell_R(M/I^{an+1}) = \sum_{i=0}^{an} \phi_{I,M}(i).$

(since $(a-1)(n+1) \leq an+1$ for $n \geq a-1$)

Def: $R$ a local Noetherian ring, $M$ finite over $R$. $\chi_{I,M}(m) = \text{length}_{R}(M/I^mM)$.

Then $d(M) = \deg \chi_{I,M}$

$\begin{cases} 0 & \ell_R(M) < \infty \\ \deg (\phi_{I,M}) + 1 & \text{otherwise} \end{cases}$

This definition of $d(M)$ is independent of choices of ideals of definition by the previous lemma: $\chi_{I,M}(n) \leq \chi_{I,M}(an) \leq \chi_{I,M}(a'n)$ $\Rightarrow$ $\deg \chi_{I,M} = \deg \chi_{I'M}$.

Remark: Why the flexibility about choosing $I$?

Example: Consider $R = k[x,y,z]/(z^2-xy)$, $M = (x,y,z)$. Then

Lemma:
If $M' \hookrightarrow M$ is of finite colength, but neither has finite length, then:
$\chi_{I,M} - \chi_{I,M'}$ has degree strictly less than the degree of either $\chi_{I,M}$ or $\chi_{I,M'}$.

Pf: By a previous lemma, $\exists C_1, C_2$ s.t.

$\begin{cases} \phi_{I,M}(n) \geq \phi_{I,M}(n-C_1-C_2) \\ \phi_{I,M'}(n) \geq \phi_{I,M}(n-C_2) \end{cases}$

Moreover, $\deg \phi_{I,M}(n) = d \geq 0$ $\Rightarrow \frac{1}{n^d} (\phi_{I,M}(n+C_2)) \geq \frac{1}{n^d} \phi_{I,M}(n) \geq \frac{1}{n^d} (\phi_{I,M}(n-C_1)-C_2)$

Letting $n \to \infty$ $\Rightarrow$ the leading coefficient of $\phi_{I,M}$ and $\phi_{I,M'}$ are the same.

$\Rightarrow \chi_{I,M} - \chi_{I,M'}$ has 0-coefficient for $n^d$.
Lemma:

\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 , \text{ s.e.s.} \]

Then:

\[ \max \{ \deg X_{i,M'}, \deg X_{i,M} \} = \deg X_{i,M}. \]

Furthermore, if \( M' \) doesn't have finite length, then

\[ \deg (X_{i,M} - X_{i,M'} - X_{i,M''}) < \deg (X_{i,M}). \]

**Pf:** By one of the previous lemmas, \( \phi_{1,M}(n) = \phi_{1,N}(n) + \phi_{1,M}(n) \) for \( n \gg 0 \) and \( M' \hookrightarrow N \) is of finite co-length. In particular, \( \phi_{1,N} \) and \( \phi_{1,M} \) have the same leading coefficient and degree. \( \Rightarrow \) \( X_{i,M}(n), X_{i,N}(n) + X_{i,M}(n) \) (which has the same leading coefficient and degree as \( X_{i,M}(n) + X_{i,M}(n) \), in any case) have the same leading coefficient and degree. The result follows. \( \square \)
§8. Dimension

Def: The Krull dimension of a ring $R$ is the Krull dimension of $R$ as a topological space, i.e. $\text{dim } R = \sup \{ n \mid \exists \beta_0 \leq \beta_1 \leq \ldots \leq \beta_n, \beta_i \in R \text{ prime} \}$

Def: The height of $\beta \in R$ is defined as $\text{dim } R_\beta$ (codim of $V(\beta)$ in $\text{Spec } R$)

E.g. $\text{dim } R = 0$ iff every prime is maximal.

Lemma: $\text{dim } R = \sup \{ \text{height } m \mid m \text{ maximal} \}$.

Lemma: $R$: Noetherian ring of $\text{dim } 0 \iff R$: Artinian.

Pf: The converse is proved already.

Now, assume $R$ is Noetherian of $\text{dim } 0 \Rightarrow \text{Spec } R$ is Noetherian $\Rightarrow \text{Spec } R = \mathbb{Z}_1 \cup \ldots \cup \mathbb{Z}_r$, a finite union of irreducible components. Now since $\mathbb{Z}_i = \mathbb{Z}(\beta_i)$ for minimal primes $\beta_i$'s and $\text{dim } R = 0$, $\beta_i$'s are maximal $\Rightarrow \text{Spec } R = \{ \beta_1, \ldots, \beta_r \}$ (every prime contains a minimal prime). Set $I = \bigcap \beta_i = \text{rad } R \Rightarrow I^N = 0$ for some $N$ since $R$ is Noetherian. $\Rightarrow R = \prod R_i$, $R_i = R_{\beta_i}$. Now $R_i$ is local, Noetherian with exactly one prime, and which is nilpotent (nilradicals localize!) $\Rightarrow \ell(R_i) < \infty \Rightarrow \ell(R) < \infty$.

Cor:

$R$: Noetherian local, then $\text{dim } R = 0 \iff \ell(R) = 0$

Pf: $\ell(R) = 0 \iff \ell_c(R) < \infty$

Upshot: $R$: Noetherian local. $\text{dim } R = 0 \iff \ell(R) = 0 \iff m = \sqrt{0} \iff \text{Artinian} \ldots$

How about $\text{dim } R = 1$?

Lemma:

$R$: Noetherian local. TFAE:
1. \( \dim R = 1 \)
2. \( d(R) = 1 \)
3. \( \exists x \in \mathfrak{m} \), \( x \) not nilpotent, s.t. \( \mathfrak{m} = \sqrt{(x)} \)
4. \( \exists \) an ideal of definition generated by 1 element but \( \mathfrak{m} \) an ideal of definition generated by 0.

Pf: 3) \( \Leftrightarrow \) 4) is easy.

1) \( \Rightarrow \) 3). Assume \( \dim R = 1 \). Let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \) be minimal primes of \( R \). Then the only other prime is \( \mathfrak{m} \). Since \( \mathfrak{m} \not\subset \mathfrak{p}_i \), by prime avoidance, \( \exists x \in \mathfrak{m}, x \not\in \mathfrak{p}_i, \forall i \). Hence the only prime containing \( x \) is \( \mathfrak{m} \) and \( x \) is not nilpotent. \( \Rightarrow \sqrt{(x)} = \mathfrak{m} \) and \( \sqrt{(x)} = \mathfrak{m} \)

3) \( \Rightarrow \) 2). Assume \( \exists x \in \mathfrak{m} \), \( x \) not nilpotent, \( \mathfrak{m} = \sqrt{(x)} \). Because \( x \) is not nilpotent \( d(R) = 0 \) (the result above). On the other hand, \( I = (x) \) is an ideal of definition, so we may compute \( d(R) = \deg(\mathfrak{p}_2, R) + 1 \). In other words, we now look at \( \mathfrak{p}_2, R(n) = \text{Length}_R \left( \frac{R}{(x^n)} \right) = \text{Length}_R \left( \frac{(x^n)}{(x^m)} \right) \). Since \( \frac{R}{(x^n)} \rightarrow \frac{(x^n)}{(x^m)} \rightarrow 0 \rightarrow \frac{l'}{l''} \), \( \Rightarrow \mathfrak{p}_2, R(n) \sim \text{Const.} \forall n \). \( \Rightarrow \deg \mathfrak{p}_2, R = 0 \Rightarrow d(R) = 1 \).

2) \( \Rightarrow \) 1). Argue by contradiction. Suppose \( \exists p \neq q \in \mathfrak{m} \) distinct primes. Pick an ideal of definition, and consider \( 0 \rightarrow p \rightarrow R \rightarrow R/p \rightarrow 0 \). By assumption, we have \( d(R) = \max \{ d(p), d(R/p) \} = 1 \). \( \Rightarrow d(R/p) \leq 1 \). Clearly \( d(R/p) \neq 0 \) since \( R/p \) is not 0-dimensional. Now, pick \( x \in q, x \not\in p \) \( 0 \rightarrow R/p \rightarrow R \rightarrow R/p \rightarrow 0 \) is exact. \( \Rightarrow \chi \rightarrow \chi \rightarrow \chi \rightarrow \chi \rightarrow 0 \) has degree \( < 1 \) \( \Rightarrow d(R/(x+p)) = 0 \Rightarrow \dim R/(x+p) = 0 \), which is absurd since \( q \not\subset \mathfrak{m} \). \( \square \)

Remark: In 2) \( \Rightarrow \) 1) we used the fact that if \( R \rightarrow R' \) and \( M' \) is an \( R' \)-module \( \Rightarrow l'(M') = l'(M) \), and if \( I \) is an ideal of definition in \( R \), then \( IR' \) is an ideal of definition in \( R' \).

More generally, we can show by doing dimension reduction to show that:
Lemma:
\( R: \) local Noetherian ring, \( d \geq 0 \). TFAE:
1. \( \dim R = d \)
2. \( \alpha(R) = d \)
3. \( \exists \) an ideal of definition generated by \( d \)-elements but \( \not\exists \) any ideal of definition generated by \( d-1 \) elements.

Cor. \( R: \) local Noetherian ring, then \( \dim R < \infty \).
Proof: We may calculate \( \dim R \) by \( \deg_X R \) (1) above, which is \( < \infty \). \( \Box \)

Warning: \( \exists \) Noetherian rings with \( \dim = \infty \).

Note also that by Nakayama’s lemma, \( \dim m, \frac{m}{m^2} = \min \# \) of generators of \( m \) as an ideal \( \Rightarrow \dim m \), by (3) above.

Def: If \( \dim R = \min \# \) of generators of \( m \), then \( R \) is called a regular local ring.

Example: \( R = (k[x,y,z]/(xy, zw))m, m = (x, y, z, w) \)
Claim: \( R/(x+y, z+w)R \) is finite.
Indeed, \( R/(x+y, z+w)R \cong (k[x,y,z]/(xy, zw, xy, zw))m \cong (k[x,z]/(x, z^2))m \cong k[x,z]/(x, z^2) \cong k \oplus kx \oplus kz \oplus kxz \) \( \cong (M = (x, z)), \) anything like \( a_0 + \Sigma a_j x^j z^j \) \( a_0 \neq 0 \) is already a unit in \( k[x,z]/(x, z^2) \). Thus \( \dim R \leq 2 \). But \( (xy, zw) \subseteq (x, z) \subseteq (x, z, y) \subseteq (x, y, z, w) \) exhibits it a length 2 chain, thus \( \dim R = 2 \).

Lemma: \( R: \) Noetherian ring.
(a) \( x \in R, \beta, \gamma \in R \) primes s.t. \( \beta \neq 0, (x) \subseteq \beta \) and \( \gamma \) is a minimal such. Then there is no prime strictly between \( \beta \) and \( \gamma \).
(b). If \( x \in R \) and \( q \) minimal over \( x \), then the height of \( q \) is either 0 or 1.

**Proof:** The primes between \( p \) and \( q \) are the primes of \( R' = (R/p)q = R_q/pR_q \).
and \( qR' \) is the maximal ideal of \( R' \), which is also minimal over \( xR' \). \( \Rightarrow \sqrt{qR'} = qR' \) and by previous lemmas, \( \dim R' \leq 1 \). \( \Rightarrow \dim R = 1 \) holds if \( xR' \neq 0 \), or \( x \in p \).
Part (b) follows from (a), by choosing any \( q \in D(x) \). \( \square \)

Lemma: \((R, m, x)\): Noetherian local ring, \( x \in m \). Then \( \dim R \leq \dim R/(ax) + 1 \).
Equality holds if \( x \in \) any minimal prime of \( R \).

**Remark:** By prime avoidance, if \( \dim R \geq 1 \), such \( x \) that \( \Rightarrow \dim R/(ax) + 1 \).

**Proof:** \( R \to R/(ax) \). Pick \( \bar{x}_1, \ldots, \bar{x}_d \in R/(ax) \) where \( d = \dim R/(ax) \) and \( (\bar{x}_1, \ldots, \bar{x}_d) \) is an ideal of definition. \( \Rightarrow R/(ax, x_1, \ldots, x_d) \subseteq R/(ax) \) has finite length. \( \Rightarrow (x, x_1, \ldots, x_d) \) is an ideal of definition. \( \Rightarrow \dim R \leq d + 1 \).
Moreover, prime chains that occur in \( R/(ax) \) are prime chains in \( R \) containing \( x \). If \( x \in \) any minimal prime, such chains can be extended at least by one of the minimal primes. \( \square \)

Two lemmas on filtrations of modules.

**Lemma 1:** \( R \): a ring, \( M \): a finite \( R \)-module. Then \( \exists \) a filtration of \( M \) by submodules \( 0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M \) s.t. \( M_i/M_{i-1} \cong R/I_i \) for some \( I_i \).

**Proof:** \( M = (x_1, \ldots, x_r) \), then just set \( M_i = (x_i, \ldots, x_i) \). \( \square \)

**Lemma 2:** \( R \): Noetherian, \( M \) finite \( R \)-module. Then \( \exists \) a filtration \( 0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \) s.t. \( M_i/M_{i-1} \cong R/\beta_i \) for some prime \( \beta_i \).

**Proof:** By lemma 1, it suffices to prove for \( M = R/I \). Consider the set of ideals \( \{J \supseteq I \} \) the conclusion fails for \( R/J \). We will show that it's empty. Otherwise, it would contain a maximal element, say \( J \), s.t. \( R/J \) has no such filtration. Clearly \( J \) cannot be prime, otherwise \( 0 \in R/J \) is a filtration. Then \( \exists a, b \in R \) s.t. \( ab \in J \) but \( a, b \notin J \). Now \( 0 \to \frac{R + J}{a} \to \frac{R}{a} \to \frac{R}{aR+J} \to 0 \) is s.e. and we have \( \frac{aR+J}{a} \cong \frac{R}{a} \), \( I \nsubseteq J \) since \( b \notin I \) \( \Rightarrow \) both \( I' \) and \( aR + J \) strictly contains \( J \) thus \( R/I' \), \( R/aR+J \) have a required filtration. \( \Rightarrow R/J \) has a required
filtration. Contradiction. □

Example: \( R = \mathbb{M} = k[x] \), we see that such filtrations are highly non-unique.
1. \( 0 \leq M \).
2. \( 0 \leq (x) \subseteq M \).
3. \( 0 \leq (x-\alpha)(x) \subseteq (x) \subseteq M \).

However, as will be shown later, the minimal primes occurring in any such a filtration will be unique with multiplicity!

Support of a module
Def: \( R \) a ring. \( M \) an \( R \)-module. \( \text{Supp}(M) \equiv \{ p \in \text{Spec} R \mid M_p \neq 0 \} \).

Lemma: If \( M \) is finite, then \( \text{Supp} M \) is closed.
Proof: We show that, \( \text{Spec} R \setminus \text{Supp} M \) is open. Indeed, let \( M = (x_1, \ldots, x_n) \).
\[ M_p = 0 \Rightarrow \sum_{i=1}^{n} f_i = 0 \text{ for } f_i \in R. \]
\[ M_p = 0 \Rightarrow \exists f_1, \ldots, f_n \text{ s.t. } f_i(x_i) = 0 \text{ in } M. \]
\[ \Rightarrow \exists f_1, \ldots, f_n \text{ s.t. } f(x_i) = 0 \text{ in } M. \]
\[ \Rightarrow M_q = 0, \forall q \in \text{Def} M. \]
(A direct proof. \( M \) is an \( R/\text{Ann} M \)-module \( \Rightarrow M_p \) is an \( (R/\text{Ann} M)_p \)-module.
\[ M_p = 0 \Rightarrow (R/\text{Ann} M)_p = 0 \Rightarrow \text{Ann} M \cdot R_p \neq R_p \Rightarrow p \geq \text{Ann} M. \]
Conversely if \( p \geq \text{Ann} M \), \( \exists x \in \text{Ann} M, x \cdot f \Rightarrow f(x) \in R_p \text{ is a unit and } f(x) \text{ is a unit and } f(x) \text{ is a unit.} \]
\[ \Rightarrow M_p = 0 \) □

Lemma: \( R \) ring. \( M \) a \( R \)-module.
1. If \( M \) is finite, then \( \text{Supp}(M/I) = \text{Supp} (M) \cap V(I) \)
2. \( N \subseteq M \Rightarrow \text{Supp} N \subseteq \text{Supp} M \)
3. \( M \rightarrow Q \Rightarrow \text{Supp} Q \subseteq \text{Supp} M \)
4. S.E.S: \( 0 \rightarrow N \rightarrow M \rightarrow Q \rightarrow 0 \Rightarrow \text{Supp} M = \text{Supp} N \cup \text{Supp} Q. \)

Proof: (2), (3), (4) follows directly from localization being exact.
For (1), since \( M \rightarrow M/I \rightarrow \text{Supp}(M/I) \subseteq \text{Supp} M \), and \( (R/I)^{\oplus n} \rightarrow M/I \rightarrow \text{Supp}(M/I) \subseteq V(I) \), where \( n \) is the number of a set of generators. Conversely, if \( M_p \neq 0 \) and \( I \subseteq p \Rightarrow M_p/I M_p \neq 0 \) by Nakayama's lemma \( \Rightarrow (M/I)_{p} \neq 0 \). □
Cor. Suppose $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, M_i/M_{i-1} \cong R/p_i$, then all $p_i \in \text{Supp}(M)$. □

Lemma: $(R, m, \kappa)$: local Noetherian ring, $M$: finite $R$-module.
Then $\text{Supp}(M) = \{m\} \iff \ell_{R}(M) < \infty$.

Pf: $\Rightarrow$: $M$ has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, M_i/M_{i-1} \cong R/m_i$ $\Rightarrow m^n M = 0 \Rightarrow \ell_R(M) < \infty$.

$\Leftarrow$: $m^n M = 0$ for some $n \in \mathbb{N}$. If $p_i \nsubseteq m$, then $\exists x \in m, x \notin p_i$. $\kappa^n M = 0$ but $\kappa^n \notin R/p_i$ is a unit $\Rightarrow M_p = 0$. □

Lemma:
$M$: an $R$-module with filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M, M_i/M_{i-1} \cong R/p_i$. Then the set of minimal elements in $\{p_i\}$ is the set of minimal elements in $\text{Supp}(M)$.
Moreover we have $\# \{ i \mid p_i = p, \ p \text{ minimal} \} = \ell_{R_p}(M_p)$.

Pf: For $M_p$, $0 = M_0 p \subseteq M_1 p \subseteq \cdots \subseteq M_r p = M_p$ is again a filtration, with $M_i p/M_{i-1} p \cong (R/p_i)p \cong R_p / p_i R_p$. If $p_i R_p$ is either $R_p$ or a prime ideal. Hence $M_p \neq 0 \iff$ at least one of the inclusions is strict $\iff p \subseteq p_i$ for some $p_i$ minimal.
If $p_i$ is minimal, the filtration $0 = M_0 p \subseteq \cdots \subseteq M_r p = M_p$ is of successive quotients either $\kappa(p_i)$ or $0 \Rightarrow \ell_{R_p}(M_p) = \# \{ \text{non-trivial inclusions} \} = \# \{ i \mid M_i/M_{i-1} \cong R/p_i \}$.

□

Lemma:
$R$: Noetherian local ring, $M$ a finite $R$-module, then $d(M) = \text{dim} \text{Supp}(M)$.

Pf: Take a filtration as above. We have $\text{dim} \text{Supp}(M) = \max \{ \text{dim} R/p_i \}$. But since $d(M) = \max \{ d(R/p_i) \}$ and $d(R/p_i) = \text{dim} R/p_i$, the result follows. □

Associated primes.
Def: $R$: Noetherian and $M$ finite. A prime $p$ is called an associated prime of the module $M$ iff $\exists m \in M, \text{Ann}(m) = p$ (or $R/p \rightarrow M$). $\text{Ass}(M) = \{ p \}$ $p$ associated primes of $M$. 

Lemma:

$R$ Noetherian and $M$ finite. Take a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ s.t. $M_i / M_{i-1} \cong R / \mathfrak{p}_i$. Then $\text{Ass}(M) = \{ \mathfrak{p}_i \}$. 
Pf: By induction on length of the filtration.

$n = 1$. $M \cong R / \mathfrak{p}$. Any non-zero element in $M$ has annihilator $\mathfrak{p}$.

Now suppose the lemma is true for modules with a filtration as above of length $\leq n-1$. Pick $m \in M$ whose annihilator is a prime $\mathfrak{p}$.

If $m \in M_{n-1}$, then we are done by induction hypothesis.

If $m \notin M_{n-1}$, then $0 \neq \overline{m} \in M_n / M_{n-1} \cong R / \mathfrak{p}_n \Rightarrow \mathfrak{p} \subseteq \text{Ann} \overline{m} = \mathfrak{p}_n$. If $\mathfrak{p} = \mathfrak{p}_n$ then we are done, otherwise $\mathfrak{p} \subseteq \mathfrak{p}_n$. Pick $f \in \mathfrak{p}_n, f \notin \mathfrak{p}$. Consider the annihilator of $f \overline{m} \in M_{n-1}: gf \overline{m} = 0 \Rightarrow gf \in \mathfrak{p}_n, f \in \mathfrak{p} \Rightarrow g \in \mathfrak{p}$. Conversely, any $g \in \mathfrak{p}$ kills $f \overline{m} \Rightarrow \text{Ann}(f \overline{m}) = \mathfrak{p}$, we win by induction. □

Cor. Condition as above, $\text{Ass}(M)$ is finite. □

Prop:

$R$ Noetherian, $M$ finite. Then the following sets of ideals coincide.

1. Minimal primes in $\text{Supp}(M)$
2. Minimal primes in $\text{Ass}(M)$
3. Minimal primes in a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$, $M_i / M_{i-1} \cong R / \mathfrak{p}_i$.

Pf: We have shown (1) = (3) already. Thus it suffices to show that, if $\mathfrak{p}$ is a (minimal) element in a filtration, then it is in (2), by the previous lemma.

Now, let $\mathfrak{p} \in \{ \mathfrak{p}_i \}$ and $i$ be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $m \in M_i, m \notin M_{i-1}$, then $\text{Ann}(m) \subseteq \mathfrak{p}_i$. Also $\mathfrak{p}_i, \cdots, \mathfrak{p}_1, \mathfrak{p}_0, m = 0 \Rightarrow \text{Ann}(m) \supseteq \mathfrak{p}_i, \cdots, \mathfrak{p}_1, \mathfrak{p}_0$. By our choice of $i$, $\mathfrak{p}_i, \cdots, \mathfrak{p}_1, \mathfrak{p}_0 \supseteq \mathfrak{p}_i \Rightarrow \exists f \in \mathfrak{p}_i, f \in \mathfrak{p}_i, \cdots, \mathfrak{p}_1$. Consider $\text{Ann}(f \overline{m})$. $f \overline{m} = 0$ and $f \overline{m} \in M_{i-1} \Rightarrow \mathfrak{p}_i \subseteq \text{Ann}(f \overline{m})$ and $\text{Ann}(f \overline{m}) \subseteq \mathfrak{p}_i \Rightarrow \text{Ann}(f \overline{m}) = \mathfrak{p}_i$. □

Lemma:

$R, M$ as above, then $\bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p} = \text{set of zero divisors on } M$. 
Pf: \( \subseteq \) is by definition.
\( \supseteq \) Pick \( x \) a zero divisor. Let \( 0 \neq N = \{ m | xm = 0 \} \). \( \Rightarrow \) \( 0 \neq \text{Ass}(N) \subseteq \text{Ass}(M) \).
Take \( n \in N \) s.t. \( \text{Ann}(n) = q \in \text{Ann}M \), since \( xn = 0 \), \( x \in q \).

Example: \( R = M = k[x, y]/(x^2, xy) \)
\( x \in M \) has annihilator \( (x, y) \), then one may guess that \( \text{Ass}(M) = \{ (x), (x, y) \} \). Indeed we can find a filtration of \( M \):
\( 0 \subseteq (x) \subseteq M \)
and \( (x) \subseteq \frac{R}{(x, y)}, \frac{M}{(x)} \subseteq \frac{R}{(x)} \) the prop. above applies.

In particular, note that although \( \text{Spec}R \cong \text{Spec}k[y] \) as topological spaces,
this coarser invariant of \( \text{Ass}(R) \) shows that \( \text{Spec}R \) is not isomorphic to \( \text{Spec}k[y] \) as schemes.
§9. Regular Sequences and Depth.

Def: \( R \): a ring, \( M \) an \( R \)-module. A sequence of elements \( f_1, \ldots, f_r \in R \) is called \( M \)-regular iff:

1. \( f_i \) is a non-zero divisor on \( M/(f_1, \ldots, f_{i-1})M \)
2. \( M/(f_1, \ldots, f_r)M \neq 0 \).

If \( I \subseteq R \) is an ideal and \( f_1, \ldots, f_r \in I \) with (1) and (2), they are called an \( M \)-regular sequence in \( I \). If \( M = R \), it’s just called a regular sequence in \( I \).

The concept is not so well-behaved in sense of the following:

Example:

1. \( R = M = k[x, y, z] \) (a ”global” ring) \( (x, y(1-x), z(1-x)) \) is then regular; but \( (y(1-x), z(1-x), x) \) is not, since in \( R/(y(1-x), z(1-x)) \), \( z(1-x) \) is a zero divisor, it kills \( y \neq 0 \).
2. \( R = M = k[x, y, w_0, w_1, \ldots]/I \), \( I = (y w_i, w_i-x w_{i+1}, \ldots) \) (Non-noetherian).

Claim: \((x, y) \) is a regular sequence, but \((y, x) \) isn’t.

Indeed \( R/(x) \cong k[y] \), but \( y \) a zero divisor in \( R \).

Def: If \( (R, m) \) is a local ring, \( M \) an \( R \)-module. Then \( \text{depth}_M(M) \cong \text{sup of length of } M \)-regular sequence in \( M \).

Lemma:

Let \( R \) be Noetherian local, \( M \) finite over \( R \). If \( (x_1, \ldots, x_k) \) is a regular sequence, then so is any permutation \( (x_{i_1}, \ldots, x_{i_k}) \).

Pf: Note that condition (2) forces \( x_i \in m \).

It suffices to prove for 2 elements, since any permutation is a combination of transpositions. Now, take \( x \in m \) a non-zero divisor, \( y \in m \) a non-zero divisor on \( M/xM \).
Consider the diagram:

\[
\begin{align*}
0 & \to M \to M \to M/xM \to 0 \\
\downarrow y & \downarrow y & \downarrow y \\
0 & \to M \to M \to M/xM \to 0 \\
\downarrow & \downarrow & \downarrow \\
M/yM & \to M/yM \to M/(xy)M \to 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & & & \\
\end{align*}
\]

\[\Rightarrow 0 \to \ker y \to \ker y \to 0 \to M/yM \to M/yM \to M/(xy)M \to 0 \text{ is exact}\]
\[\Rightarrow m \ker y = \ker y (y \in m) \text{ and } M/yM \to M/yM \text{ is an injection}\]
\[\Rightarrow \ker y = 0 \text{ by Nakayama and } x \text{ is a non-zero divisor on } M\]
\[\Rightarrow (y, x) \text{ is regular}.\] □

**Lemma:**

- **R:** Noetherian local, **M:** finite \( R \)-module, \( I \subseteq m \) an ideal of \( R \). TFAE:
  1. \( \exists x \in I \) which is not a zero-divisor on \( M \), i.e. \( \text{depth}_M M \gg 1 \)
  2. We have \( I \nsubseteq q \) for all \( q \in \text{Ass}(M) \), (in particular \( m \in \text{Ass}(M) \)).

**Pf:** We know that \( U_{q \in \text{Ass}(M)} q = \{ \text{zero-divisors on } M \} \)

(2) \( \Rightarrow \) (1) Since \( \text{Ass} M \) is a finite set, by prime avoidance, \( \exists x \in I. x \in q \forall q \in \text{Ass} M \)

(1) \( \Rightarrow \) (2) Obvious □

**Cor.** In the previous lemma, if \( I=m \), \( \text{depth}_M \gg 1 \iff m \in \text{Ass} M \) □

**Cor.** If \( R \) Noetherian local, with residue field \( k \), then \( \text{depth}_R = 0 \) iff there exists a non-zero module map \( k \hookrightarrow R \).

**Pf:** \( \text{depth}_R = 0 \iff m \in \text{Ass} M \iff x \hookrightarrow R \). □

**Example:** \( R = k[x,y]/(xy, x^2) \), then \( x \neq 0 \) and \( mx = 0 \Rightarrow \text{depth} R = 0 \)
Lemma:

$R$: Noetherian local, $M$: finite $R$-module. $\text{depth}_R M \leq \dim(\text{Supp}(M)) = d(M)$.

Pf: If $M$ is of finite length, then $m \in \text{Ass}(M)$, and $\text{depth} M = 0$.

Now if $M$ is not of finite length, and take any regular sequence on $M$, say $f_1, \ldots, f_r \in M$, we will show that its length $r \leq d(M)$ by induction on $d(M)$. The $d(M) = 0$ case is done by the first sentence.

Since $(f_2, \ldots, f_r)$ is a regular sequence on $M/(f_1)M$, $r-1 \leq \text{depth}(M/(f_1)M)$. It suffices to show that $d(M/(f_1)M) \leq \dim M - 1$, then $r-1 \leq \text{depth}(M/(f_1)M) \leq d(M/(f_1)M) \leq d(M) - 1$.

Since $0 \to M \xrightarrow{f_1} M \to M/(f_1)M \to 0$ is exact, by the last lemma of §7, we know that $d(M/(f_1)M) = \deg \chi_{M/(f_1)M} < \deg \chi_M = d(M)$. □

Lemma:

$R$ a ring and $J = (f_1, \ldots, f_c)$ generated by a regular sequence, then the graded ring $\oplus_{n \geq 0} J^n/J^{n+1}$ is graded isomorphic to $R[J[x_1, \ldots, x_c]]$.

(A variant for modules also holds if $f_1, \ldots, f_c$ is a regular sequence on $M$, in which case the graded module $\oplus_{n \geq 0} J^n M/J^{n+1}M$ is graded isomorphic to $M/JM [x_1, \ldots, x_c]$).

Spec $\oplus_{n \geq 0} J^n/J^{n+1} \to \text{Spec } R[J]$ is called the normal cone of Spec $R/J$ in Spec $R$. c.f. the example below.

Example: $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3 + x^{100} + z^{100} + y^{1000})$, $J = (x, y, z)$

Then $J/J^2 \cong \mathbb{C}^3 = \mathbb{C}\{x, y, z\}$. Surely $\Phi_3: \text{Sym}^n(J/J^2) \to J^n/J^{n+1}$.

However, $x^3 + y^3 + z^3 \in \text{Ker } \Phi_3$ since $x^{100} + z^{100} + y^{1000} \in J^4 \Rightarrow R[J[x, y, z] \not\cong \oplus_{n \geq 0} J^n/J^{n+1}$, and $(x, y, z)$ is not a regular sequence.

In fact $\oplus_{n \geq 0} J^n/J^{n+1} \cong \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$ and its max-spectrum is a cone in $\mathbb{C}^3$.

Proof of lemma.

We prove by induction on $c$. $c=1$ is clear.

Goal: $\sum_{|I|=n} a_I f_I \in J^{n+1} \iff a_I \in J$, $\forall I = (i_1, \ldots, i_c) \in \mathbb{Z}^c$. 

We can write any element in $J_+^m$ as $\sum_{|I|=n} b_{i_1} f_{i_1}^2$ with $b_{i_1} \in J$. Thus after substitution we are reduced to prove if $a = \sum_{|I|=n} a_{i_1} f_{i_1}^2 = 0$ then $a_{i_1} \in J$, $\forall I$. Collecting terms, we have $a = \sum_{|I|=n} \left( \sum_{i_1 \in I} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i}$. Now we shall show by ascending induction on $0 \leq e \leq n$ that if we have

$$\sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} = 0$$

then $a_{i_1} \in J$. Set $J' = (f_1, \ldots, f_{c-1})$. $L = 0$ is the induction hypothesis of $c - 1$. Now $f_c \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} = -\sum_{|I|=n} a_{i_1} e_i f_{i_1}^2 \equiv 0 \mod J'$ and $f_c$ is a non-zero divisor $\Rightarrow \sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} = 0$.

Collecting terms, we have $a = \sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i}$. We will prove by induction that $\sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} \in J_+^{e+1}$, then $a_{i_1} \in J$.

Now $\sum_{|I|=l} a_{i_1} f_{i_1}^2 \in J_+^{e+1} \mod (f_l)$, and since $(f_1, \ldots, f_{c-1})$ is a regular sequence mod $(f_1)$, by induction on the length of sequence, $a_{i_1} f_1 \in J \Rightarrow a_{i_1} f_1 \in J$. It follows that $\sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} \in J_+^{e+1} \Rightarrow f_1 \left( \sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} \right) \in J_+^{e+1}$.

$$D = a_{i_1} f_1 + a_{i_1} f_1 + \cdots + a_{i_1} f_c \mod J \Rightarrow \sum_{|I|=l} a_{i_1} e_i f_{i_1}^2 = 0 \Rightarrow a_{i_1} \in J$$

$$\sum_{|I|=l} a_{i_1} e_i f_{i_1}^2 = 0 \Rightarrow a_{i_1} \in J$$

$$\Rightarrow a_{i_1} \in J_+^{e+1} \Rightarrow a_{i_1} f_1 + \cdots + a_{i_1} f_c \mod J_+^{e+1}$$

$$\Rightarrow a_{i_1} = b_{i_1} f_1 + \cdots + b_{i_1} f_c \mod J_+^{e+1}$$

$$\left( \sum_{e=0}^n \left( \sum_{|I|=n-e} a_{i_1} e_i f_{i_1}^2 \right) f_{i_1}^{e_i} \right) = 0$$
• Digression: Homological Algebra

Lemma: $R$ a ring, $M$ an $R$-module.

1. There exists an exact complex $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$, where $F_i$ are free $R$-modules.

2. If $R$ is Noetherian and $M$ finite, then we can choose $F_i$ finite free. □

Notation: We will denote the complex $\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots$ by $F_i$.

1. The $i$-th homology group is defined as $H_i(F_i) = \ker d_i / \text{Im} d_{i+1}$.

2. Morphisms of complexes $\alpha: F \to G$ is given by maps $\alpha_i: F_i \to G_i$ such that

\[
\begin{array}{ccc}
\cdots & \xrightarrow{d_i} & F_i \\
\downarrow{\alpha_i} & & \downarrow{\alpha_{i-1}} \\
\cdots & \xrightarrow{d_i} & G_i
\end{array}
\]

and $\alpha$ induces maps on homology groups: $H_i(\alpha): H_i(F_i) \to H_i(G_i)$

3. $\alpha, \beta: F \to G$: morphisms of complexes, then a homotopy between $\alpha$ and $\beta$ is a collection of maps $F_i \to G_{i+1}$ s.t. $\alpha - \beta = d_{i+1} \circ h_i + h_{i-1} \circ d_i$

\[
\begin{array}{ccc}
\cdots & \xrightarrow{d_{i+1}} & F_i \\
\downarrow{h_i} & & \downarrow{h_{i-1}} \\
\cdots & \xrightarrow{d_i} & F_{i-1}
\end{array}
\]

Immediate consequence: $\alpha \simeq \beta \Rightarrow H_i(\alpha) = H_i(\beta)$ on homology groups.

4. Cohomological notation: $F^\ast: \cdots \to F^i \xrightarrow{d^i} F^{i-1} \to \cdots$, $H^i(F^\ast) = \ker d^i / \text{im} d^i$

Lemma: Suppose $G_\ast \to N \to 0$ is an exact resolution, $F_\ast \to M \to 0$ is a free resolution, and $\gamma: M \to N$ is an $R$-map, then:

1. $\exists \alpha: F_\ast \to G_\ast$ which induces $\gamma$, i.e. $H_0(\alpha) = \gamma$.

2. Any two such maps are homotopic. □

Def: $(\text{Ext}_R(M,N))$ Choose a free resolution $F_\ast$ of $M$. Consider the complex: $\text{Hom}_R(F_\ast, N): 0 \to \text{Hom}_R(F_0, N) \xrightarrow{d_1^1} \text{Hom}_R(F_1, N) \xrightarrow{d_2^1} \text{Hom}_R(F_2, N) \to \cdots$

Set: $\text{Ext}^i(M,N) = H^i(\text{Hom}(F_\ast, N))$. It's "the" concept by the next lemma:
Lemma: \( R \) a ring, \( M_1, M_2, N \) are \( R \)-modules. \( F \) a free resolution of \( M_1 \), \( G \) a free resolution of \( M_2 \). \( \varphi: M_1 \rightarrow M_2 \), and \( \alpha: F \rightarrow G \) any map of complexes s.t. \( H_0(\alpha) = \varphi \). Then the induced maps:

\[
H^i(\alpha): H^i(\text{Hom}_R(F, N)) \rightarrow H^i(\text{Hom}_R(G, N))
\]

are independent of choices of \( \alpha \). Furthermore, if \( \varphi \) is an isomorphism, so is each \( H^i(\alpha) \); if \( M_1 = M_2 \), \( \varphi = \text{id} \), so is \( H^i(\alpha) = \text{id} \). □

Rmk: This lemma says that \( \text{Ext}^i(M, N) \) form a contravariant functor in the variable \( M \).

Lemma: \( \text{Ext}^i_R(M, N) \cong \text{Hom}_R(M, N) \). □

Lemma: \( M \rightarrow M' \) gives \( \text{Ext}^i(M', N) \rightarrow \text{Ext}^i(M, N) ; N \rightarrow N' \) gives \( \text{Ext}^i_R(M, N) \rightarrow \text{Ext}^i_R(M, N') \). Thus \( \text{Ext}^i_R(M, N) \cong (\text{Mod}_R)^{\text{op}} \times (\text{Mod}_R) \rightarrow \text{Mod}_R \). □

Lemma: \( R \) a ring, \( M \) an \( R \)-module. \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) s.e.s. of \( R \)-modules \( \Rightarrow \exists \) long exact sequence:

\[
0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N'') \rightarrow \text{Ext}^1_R(M, N') \rightarrow \cdots \rightarrow \text{Ext}^i_R(M, N) \rightarrow \text{Ext}^i_R(M, N'') \rightarrow \text{Ext}^{i+1}_R(M, N') \rightarrow \cdots
\]

Example:

a). \( \text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = ? \)

\( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0 \) is a free resolution

\( \Rightarrow \) The homology of: \( 0 \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \rightarrow 0 \) computes \( \text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) \), which is the same sequence as \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \)

\( \Rightarrow \text{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) = 0 \), \( \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) = \mathbb{Z}/p \), and the higher groups \( \text{Ext}^i(\mathbb{Z}/p, \mathbb{Z}) = 0, \ i > 1 \).
b. \( R = k \left[ e_1 \right] \cong k \left[ x_1 / x_2 \right] \), i.e. \( e_2 = 0 \). \( \text{Ext}_i^k (k, k) = ? \)

F. \( \colon \quad \cdots \to R \xrightarrow{e} R \xrightarrow{e} R \xrightarrow{e} R \to k \to 0 \)

\[ \Rightarrow \text{Hom}_R (F, k) : \quad 0 \to k \xrightarrow{a} k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} \cdots \]

\[ \Rightarrow \text{Ext}_i^k (k, k) = k \text{ for all } i > 0. \]

Lemma: \( R \) a ring, \( M, N \) \( R \)-modules, \( x \in R \) s.t. either \( xM = 0 \) or \( xN = 0 \). Then \( x \) annihilates each \( \text{Ext}_i^k (M, N) \).

The two follows immediately once one realizes that for all \( x \in R \), multiplication on \( \text{Ext}_i^k (M, N) \) is the same as the map induced by \( M \xrightarrow{x} M \) or \( N \xrightarrow{x} N \) by functoriality.

Using this language, we have a next interpretation of depth:

Lemma: \( R \) Noetherian, local, with maximal ideal \( m \). \( M \) a finite \( R \)-module. \( M \neq 0 \). Then \( \text{depth}_R (M) = \min \{ i \mid \text{Ext}_i^R (R/m, M) \neq 0 \} \).

Proof: By a previous lemma, \( \text{depth}_R (M) = 0 \iff 0 \neq \text{Hom}_R (R/m, M) = \text{Ext}^0_R (R/m, M) \).

Thus we may assume that \( \text{depth}_R (M) > 0 \), and prove by induction. Now, \( \exists x \in M \) non-zero divisor on \( M \) s.t. \( \text{depth}_R (M/xM) = \text{depth}_R (M) - 1 \). (Just take \( x = f_i \), where \( f_1, \ldots, f_d \) is a maximal regular sequence.) Consider the s.e.s.

\[ 0 \to M \xrightarrow{x} M \to M/xM \to 0 \]

\( \Rightarrow \) l.e.s. \( 0 \to \text{Hom}(x, M) \xrightarrow{a} \text{Hom}(x, M) \to \text{Hom}(k, M/xM) \)

\[ \to \text{Ext}^1(k, M) \to \text{Ext}^1(k, M) \to \text{Ext}^1(k, M/xM) \]

\[ \to \ldots \to \text{Ext}^{d_1}(k, M/xM) \]

\[ \to \text{Ext}^{d_1}(k, M) \to \text{Ext}^{d_1}(k, M/xM) \to \ldots \]

By induction hypothesis, \( \text{Ext}^i(k, M/xM) = 0 \) i.e. \( d_1 = 0 \) and \( \text{Ext}^{d_1}(k, M/xM) 
eq 0 \)

l.e.s. \( \Rightarrow \text{Ext}^i(k, M) = 0 \) i.e. \( d_1 = 0 \) and \( 0 \neq \text{Ext}^{d_1}(k, M/xM) \cong \text{Ext}^{d_1}(k, M) \)

This kind of "dimension shifting" is constantly see in AG and CA.
Cor. $R$: Noetherian local ring, $0 \to N' \to N \to N'' \to 0$ a s.e.s. of finite $R$-modules. Then:

1. $\text{depth}_R N'' \geq \min \{ \text{depth}_R N, \text{depth}_R N' - 1 \}$
2. $\text{depth}_R N' \geq \min \{ \text{depth}_R N, \text{depth}_R N'' + 1 \}$

Pf: The s.e.s gives rise to a l.e.s:

$$\cdots \to \text{Ext}^i(k, N'') \to \text{Ext}^i(k, N') \to \text{Ext}^i(k, N) \to \text{Ext}^i(k, N'') \to \cdots$$

Thus, we can see that as long as $\text{Ext}^i(k, N) = 0 = \text{Ext}^{i+1}(k, N')$, $\text{Ext}^i(k, N'') = 0$; and as long as $\text{Ext}^i(k, N) = 0 = \text{Ext}^{i+1}(k, N'')$, $\text{Ext}^i(k, N') = 0$. The result follows from the previous lemma \qed

In the next section, we will see that higher depth means "better". This cor. shows that the modules in a resolution are "better" going to the left.

Rmk: We defined depth by taking $\text{Ext}^i_R(M, N) = H^i(\text{Hom}(F, N))$ for a projective (free) resolution of $M: F_\cdot \to M \to 0$. However, it can be defined by taking an injective resolution of $N: 0 \to N \to I^\cdot$, and $\text{Ext}^i_R(M, N) = H^i(\text{Hom}(M, I^\cdot))$.

One can prove that these definition agree by showing that both are equal to $H^i(\text{Tor}(\text{Hom}(F, I^\cdot)))$. 
\section*{10. Cohn-Macaulay Modules}

**Def.** $R$: Noetherian local ring. $M$ a finite $R$-module. We say that $M$ is Cohn-Macaulay if $\dim \text{Supp } M = \text{depth } M$.

**Notation:** In situation of the def., let $f_1, \ldots, f_d$ be an $M$-regular sequence with $d = \dim (\text{Supp } M) = \text{depth } (M)$. Then we say that $g \in M$ is good w.r.t. $(M; f_1, \ldots, f_d)$ if for all $i = 0, 1, \ldots, d-1$, we have $\dim (\text{Supp } M \cap V(g, f_i, \ldots, f_d)) = \dim (\text{Supp } M / (g, f_i, \ldots, f_d) M)$ $= d - i - 1$.

**Lemma:** In the situation as above
\begin{enumerate}
  \item $g$ is a non-zero divisor.
  \item $M / gM$ is CM with a maximal regular sequence $(f_1, \ldots, f_{d-1})$.
\end{enumerate}

**Pf:** By induction on $d$ ($d > 0$).

$d = 1$. Only need to show that $g$ is a non-zero divisor. Since by assumption of $g$ being good, $\dim (\text{Supp } M \cap V(g)) = 0$, and $\dim 0$ modules are trivially CM.

Let $K = \text{ker}(M \xrightarrow{g} M)$. $\text{Supp } K \subseteq \text{Supp } M \cap V(g_0)$. and $\dim K = 0$. If $K \neq 0$, then we have $\{m\} = \text{Ass } K \subseteq \text{Ass } M \Rightarrow \text{depth } M = 0$. Contradiction.

$d > 1$. Consider the diagram: (still let $\text{ker}(M \xrightarrow{g} M) = K$, $\text{ker}(M / gM \xrightarrow{g} M / gM) = K_1$)

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow g & & \downarrow g \\
0 & \to & M / gM
\end{array}
\]

Snake lemma $\Rightarrow 0 \to K \xrightarrow{f_1} K \xrightarrow{g} K_1 \xrightarrow{g} M / gM \xrightarrow{f_1} M / gM \xrightarrow{g} M / gM \xrightarrow{g} 0$. By assumption $g$ is good w.r.t. $M / gM$ (which is still CM), and the regular sequence $f_2, \ldots, f_d$. Ind. hypo $\Rightarrow K_1 = 0$ and $K \xrightarrow{f_2} K$. Nakayama $\Rightarrow K = 0$ and $g$ is a non-zero divisor. Moreover $0 \to M / gM \xrightarrow{g} M / gM$ says that $f_1$ is a non-zero divisor on $M / gM$. Again by ind. hypo. $M / (g, f_1) M$ is CM with regular sequence $f_2, \ldots, f_{d-1}$. Hence $M / gM$ is CM with a regular sequence $f_1, \ldots, f_{d-1}$. \qed
From the proof, we see that CM allows us to cut dimension down and do induction. This argument will appear often.

**Lemma:** R Noetherian local, M: CM over R. If \( g \in \mathfrak{m} \) s.t. \( \dim \text{Supp} M/gM = \dim \text{Supp} M - 1 \). Then:
1. \( g \) is a non-zero divisor
2. \( M/gM \) is CM of depth 1 less.

**Pf:** Pick a maximal regular sequence \( f_1, \ldots, f_d \) (d\( \geq \)1 by assumption, and the lemma is always true if d\( = \)1, by the proof of the previous lemma. \( g \) is a non-zero divisor, and since \( \dim \text{Supp} M/gM = 0 \Rightarrow CM \). Thus we may start with d\( \geq \)2.

If \( g \) is good w.r.t. \( (M, f_1, \ldots, f_d) \) then we are done by the previous lemma.

Now pick \( h \in \mathfrak{m} \) s.t. 1. \( h \) is good w.r.t. \( (M, f_1, \ldots, f_d) \)
2. \( \dim \text{Supp} M \cap \text{V}(h, g) = d - 2 \)

This is possible because if \( \{q_i\} \) is the set of minimal primes of \( \text{Supp} M \), \( \text{Supp} M/gM \) and \( \text{Supp} M/(c_i, \ldots, f_d)M, i = 1, \ldots, d - 1 \), then \( q_i \neq \mathfrak{m} \) since none of these modules are Artinian, and prime avoidance applies to provide an \( h \).

It follows from the previous lemma that \( (h, f_1, \ldots, f_d) \) is a new regular sequence and \( g \) is good w.r.t. \( (M, h) \). (i.e. \( g \) cuts dimension down on \( M/hM \)). Now we repeat this process, to obtain an \( h' \) s.t. 1. \( h' \) is good w.r.t. \( (M, h, f_1, \ldots, f_d) \)
2. \( \dim \text{Supp} M \cap \text{V}(h, h, g) = d - 3 \)

... (d-1) times later...

We replace \( (f_1, \ldots, f_d) \) by another regular sequence w.r.t. which \( g \) is good. Now the previous lemma applies.

A non-constructive proof: ...(after obtaining \( h \))

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow g & & \downarrow g \\
M/hM & \to & 0
\end{array}
\]

Now we finish the proof by induction on \( d \). Since \( M/hM \) is CM of \( \dim d - 1 \)
and \( \dim M/(g,h)M = d-2 \). Ind. hypo. \( g \) is a non-zero divisor on \( M/hM \) and \( M/(g,h)M \) is CM of dim \( d-2 \).

Snake lemma \( 0 \rightarrow K \overset{g}{\rightarrow} K \rightarrow 0 \) and \( 0 \rightarrow M/gM \overset{h}{\rightarrow} M/gM \rightarrow M/(g,h)M \rightarrow 0 \)

\( \Rightarrow K = 0 \) by Nakayama and \( M/gM \) is CM of dim \( d-1 \) as \( h \) is a non-zero divisor on \( M/gM \).

\( \square \)

de Jong likes the constructional way better, "the right way of doing things ..."

By this lemma, if we can take a sequence of elements that really cut down dimensions, then:

Cor. \( R \) Noetherian local. \( M \) CM over \( R \). If \( \exists g_1, \ldots, g_c \) s.t. \( \dim M/(g_1,\ldots,g_c)M = \dim M-c \), then \( g_1,\ldots,g_c \) is a regular sequence and can be extended to a maximal regular sequence.

\( \square \)

Def. A Noetherian local ring is called CM if it's CM as a module over itself.

Lemma: If \( R \) is local Noetherian CM of dim \( d \), then any maximal chain of primes \( \beta_0 \subseteq \cdots \subseteq \beta_n \) has length \( n=d \).

Pf: If \( \dim R = 0 \), the result is true trivially. Now we prove by induction.

Here \( \beta_0 \in \{q \mid \text{a minimal prime of } R\} = \{q_1, \ldots, q_r\} \). By assumption \( \beta_0 \notin \{q_1, \ldots, q_r\} \) and prime avoidance \( \Rightarrow \exists x \in \beta_0, x \notin q_i \).

Hence \( \dim R/\beta_0R = d-1 \) and thus is CM by the previous lemma. Ind. hypo \( \Rightarrow \) the maximal chain \( \beta_0/(\alpha x) \subseteq \cdots \subseteq \beta_n/(\alpha x) \) has length \( n-1 = d-1 \). \( \Rightarrow n = d \).

\( \square \)

Application: \( k[x_1, \ldots, x_d] \) has dimension \( d \). (\( k=\bar{k} \))

Indeed \( k[x_1, \ldots, x_d] \) is CM of dim \( d \) (\( x_1, \ldots, x_d \) is a regular sequence).

\( \Rightarrow \) Every maximal chain of prime ideals in \( k[x_1, \ldots, x_d] \) has length \( d \).

Now any maximal chain of prime ideals ends with \( (x_1-a_1, \ldots, x_d-a_d) \), which is isomorphic to \( (x_1, \ldots, x_d) \) by translation.
Every maximal chain of prime ideals in $k[x_1, \ldots, x_n]$ has length $d$. 
or $\text{Spec}k[x_1, \ldots, x_n]$ is equidimensional of dim $d$.

Lemma:
R local Noetherian CM ring, $\mathfrak{p} \subseteq R$ prime, $\dim R = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$.
Pf: Pick $\mathfrak{q}_0 \subseteq \ldots \subseteq \mathfrak{q}_s \subseteq \mathfrak{p}$, $\mathfrak{p}_0 \subseteq \ldots \subseteq \mathfrak{p}_s \subseteq R/\mathfrak{p}$ maximal chains of primes.
Then automatically, $\mathfrak{q}_0 = \mathfrak{p} \mathfrak{p}_0$, $\mathfrak{p}_0 = (0)$
$\Rightarrow \mathfrak{q}_0 \cap R \subseteq \ldots \subseteq \mathfrak{q}_s \cap R = \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_s$ is a maximal chain of primes in $R$. By the previous lemma, $\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} = r + s = n = \dim R$. □

Caution: This result even may not hold for rings that are localizations of finite type rings over $C$! Reason: $\text{Spec}R$ needs not be equidimensional.

Lemma:
R local Noetherian CM ring, $\mathfrak{p} \subseteq R$ prime $\Rightarrow R_{\mathfrak{p}}$ CM.
Pf: Let $c = \dim R_{\mathfrak{p}}$. We want to construct a regular sequence of length $c$ in $R_{\mathfrak{p}}$ (Note that it suffices to check for $c > 0$; $c = 0 \Rightarrow$ Artinian, which is automatically CM).
For this purpose, it’s good enough to find $f_1, \ldots, f_c \in \mathfrak{p}$, an $R$-regular sequence in $\mathfrak{p}$. Indeed, since localizations are exact:
$$0 \to R \xrightarrow{f_1} R \xrightarrow{f_2} \cdots \xrightarrow{f_c} R_{\mathfrak{p}} \xrightarrow{f_{c+1}} \cdots$$
Now $R$ CM, for $f_i$ in $\mathfrak{p}$ to be a non-zero divisor, it’s necessary and sufficient that $f_i \in \mathfrak{p}$ is any minimal prime of $R$. This is possible since $c > 0 \Rightarrow \mathfrak{p}$ is not any prime ideal $\Rightarrow \exists f_i \in \mathfrak{p}$, $f_i \in \mathfrak{p}_i$, $\forall i$ minimal prime in $R$.
Next $R/f_iR$ is again a CM-module, if $\dim(R/f_iR) = 0$ then we are done. Otherwise, it’s necessary and sufficient that $f_2$ is any minimal prime in $\text{V}(f_i)$ (By the previous lemma, all such minimal primes have $\dim R_{\mathfrak{q}} = 1$ thus $\mathfrak{p} \subseteq \mathfrak{q}$ for all these $\mathfrak{q}$’s). Such an $f_2$ exists by prime avoidance.
Repeat this argument c times.

Def: A Noetherian ring $R$ is called Cohn-Macaulay if all of its local rings are CM.

Rmk: 1) Any localization of a CM ring is CM.

2) In general, a finitely generated algebra over a CM ring need not be CM. For example $k$: a field $\Rightarrow$ CM, but $k[x,y]/(x^2, xy)$ is not CM.

Lemma: $R$ CM $\Rightarrow R[x]/CM$.

Rmk: In general, $\dim R[x] = \dim R + 1$ for Noetherian rings.

Pf: Pick $q \in R[x]$ a prime over $\mathfrak{p} \subseteq R$, i.e. $\mathfrak{p} = q \cap R$. Pick a regular sequence $f_1, ..., f_d \in \mathfrak{p}$.

Claim: $f_1, ..., f_d$ are regular in $R[x]/q$.

Indeed, $R[x]/q$ is free over $R$ as an $R$-module thus similar as in the previous lemma. $f_1, ..., f_d$ are regular in $R[x]$

$\Rightarrow f_1, ..., f_d$ are $R[x]/q$ regular since $R[x]/q$ is a further localization of $R[x]$.

(This also follows from a general fact that $R \rightarrow R[x]/q$ is flat, and under flat morphisms, regular sequences $\mapsto$ regular sequences.)

Now, consider $R[x]/q/(f_1, ..., f_d)R[x]/q \cong (R/(f_1, ..., f_d)[x])\bar{q}$, where $\bar{q}$ is the image of $q$ in $R/(f_1, ..., f_d)[x]$. It suffices to show that, for Artinian local $R'$

$R/(f_1, ..., f_d)$, $R'[x]/q$ is CM for any $q \in R'[x]$. This is the content of the next lemma. In both cases of the lemma, $f_1, ..., f_d$ can be completed into a regular sequence and $R/(f_1, ..., f_d)[x]/q$ or $R/(f_1, ..., f_d)[x]/q$ are Artinian, thus of dim 0.

$\Box$

Lemma:

$R$: Artinian local ring, then $\forall q$ prime in $R[x]$, $R[x]/q$ is CM.

Pf: There are two cases to consider. (i) $q$ contains a monic polynomial

(ii) $q$ doesn't contain any monic polynomial.
In case (i) \( f \) is a non-zero divisor and \( R[x]/(cf) \) is finite free over \( R \), so that \( R[x]/(cf) \) is Artinian, thus CM of dim 0. Hence \( R[x]/q \) is CM of dim 1.

(ii). In this case \( q = mR[x] \). Indeed \( q \cap R = m \Rightarrow k(m)[x] \rightarrow R[x]/q \), which is a quotient of \( k(m)[x] \). If \( R[x]/q \neq k(m)[x] \) then \( R[x]/q \cong k(m)[x]/(f) \) an irreducible polynomial in \( k(m)[x] \). \( f = a_0x^{m_0} + a_1x^{m_1} + \cdots + a_{N+1}x^{m_{N+1}} \) a.e. \( m_i \geq N+1 \).

But since \( m \leq q \), we may well remove all the \( a_{N+1} \) since \( R[x] \in q \), \( m \geq k \geq N+1 \) and we end up with a monic polynomial. Contradiction. Thus \( f = 0 \) and \( q = mR[x] \). Moreover, \( q \) consists of nilpotent elements \( \Rightarrow \text{Spec } R[x] \cong \text{Spec } R[x]/q = \text{Spec } k(m)[x] \) and is irreducible. Hence \( q \) is minimal and \( \dim R[x]/q = 0 \), thus CM.

Lemma: \( k \) (a field) is CM. \( \mathbb{Z} \) is CM.

Pf: \( k \) is Artinian; \( \mathbb{Z} \) is dim 1, and has \( \mathfrak{p} \) regular.

Lemma: \( R[x_1, \ldots, x_n] \) and \( \mathbb{Z}[x_1, \ldots, x_n] \) are CM.

Lemma: (Dimension Shifting)

\( R \): Noetherian local ring. CM of dim \( d \), \( M \): finite \( R \) module. Then for any s.e.s. \( 0 \rightarrow K \rightarrow R^m \rightarrow M \rightarrow 0 \), we have either \( \text{depth } K \geq \text{depth } M \) or \( \text{depth } M = d \).

Pf: By a previous lemma, \( \text{depth } K \geq \min \{ \text{depth } R, \text{depth } M+1 \} \).

Def: \( R \): Noetherian local CM ring. A finite module \( M/R \) is called maximal CM iff \( \text{depth } M = \text{depth } R \).

Lemma:

\( R \): Noetherian local, CM of dim \( d \), \( M \) any finite module. Then \( \exists \) an exact complex \( 0 \rightarrow K \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \) with \( F_j \) free and \( K \) MCM.

More precisely, if \( \text{depth } M = c \), then \( i = d-c-1 \).
This lemma is useful if we want to know whether $K_0(R) \to K_0(R') \to 0$. We may start by resolving $M$'s with projective (free) resolutions, and thus only need to worry about MCM modules and check under what conditions they are free. (regular $R$, Same)

Catenary Rings
Def: A ring is said to be catenary if $\forall$ primes $\mathfrak{p} \subseteq \mathfrak{q}$, all maximal chains of primes $\mathfrak{p} = \mathfrak{p}_0 \subseteq \ldots \subseteq \mathfrak{p}_e = \mathfrak{q}$ have the same finite length.

Lemma:
Any localization of a catenary ring is catenary. The same holds for quotient. □

⚠️ Caution: $\exists$ example of a Noetherian catenary ring $R$ and a finite type $R$-algebra which is not catenary, i.e. $\exists R[x_1, \ldots, x_n]$ not catenary, by the previous lemma.

Def: $R$ is called universally catenary if $R[x_1, \ldots, x_n]$ are catenary, $\forall c \geq 0$. Then it follows from the previous lemma that any $R$-algebra is catenary and thus converse is trivially true.

Lemma:
A Noetherian CM ring is u.c. Thus $\mathbb{Z}$, $\mathbb{R}$ are u.c.

Pf: $R[x_1, \ldots, x_n]$ is CM if $R$ is. The result follows by localizing at $\mathfrak{q}$ and the lemma that any CM local ring has maximal chains of primes equal length □
§11. Regular Rings

Def: \( (R, m, \mathfrak{k}) \) a Noetherian local ring is called regular iff \( \dim m/m^2 = \dim R \).

Lemma:
If \( (R, m, \mathfrak{k}) \) is regular, then the graded ring \( \oplus_{n \geq 0} m^n/m^{n+1} \) is isomorphic to the graded polynomial ring \( k[x_1, \ldots, x_d] \), where \( d = \dim R \). The converse is trivially true.

Pf: Let \( x_1, \ldots, x_d \) be a minimal set of generators of the ideal \( m \). Then there exists a surjection \( k[x_1, \ldots, x_d] \twoheadrightarrow \oplus_{n \geq 0} m^n/m^{n+1} \), \( x_i \mapsto x_i \in m/m^2 \).

Thus it suffices to check injectivity. \( \dim R = d \Rightarrow (n \mapsto \dim m^n/m^{n+1}) \) is a numerical polynomial of deg \( d-1 \). Meanwhile, the numerical polynomial for the polynomial ring \( k[x_1, \ldots, x_d] \) is \( q_{x_1} = \binom{m+d-1}{d-1} \). If \( a \neq 0 \), homogeneous of degree \( c \), is in the kernel, then \( k[x_1, \ldots, x_d]/\langle a \rangle \rightarrow \oplus_{n \geq 0} m^n/m^{n+1} \), but the numerical polynomial for \( k[x_1, \ldots, x_d]/\langle a \rangle \) is \( \binom{m+d-1}{d-1} - \binom{m+c+d-1}{d-1} \), which has degree \( d-2 \), contradiction. It follows that the map must be an isomorphism. \( \square \)

Lemma:
Any regular local ring is a domain.

Pf: If not, take \( 0 \neq a, b \in R \), \( ab = 0 \). Consider the filtration \( R = m \supseteq m^2 \supseteq \cdots \)
Take the smallest \( i \) so that \( a \in m_i \), \( a \notin m^{i+1} \), \( k \) so that \( b \in m_k \), \( b \notin m^{k+1} \).

(such \( i, k \) exist by Artin- Rees lemma).
\( \Rightarrow 0 \neq \bar{a} \in m_i/m^{i+1}, 0 \neq \bar{b} \in m_k/m^{k+1} \). But \( ab = 0 \Rightarrow ab \equiv 0 \mod m^{k+i+1} \)
\( \Rightarrow \bar{a} \cdot \bar{b} = 0 \), contradiction with \( \oplus_{n \geq 0} m^n/m^{n+1} \) being a domain. \( \square \)

In fact, this lemma easily generalizes to the case where \( R \) Noetherian and \( \oplus_{n \geq 0} m^n/m^{n+1} \) is a domain (e.g. \( R \) CM).

Lemma.
\( R \) regular local ring, \( \{x_1, \ldots, x_d\} \) a minimal set of generators of \( m \), (a regular system of parameters). Then \( \{x_1, \ldots, x_d\} \) is a regular sequence, and \( R/\langle x_1, \ldots, x_d \rangle \)
is a regular local ring of dim $d-c$. In particular, $R$ is CM.

Pf: $x_i$ is a non-zero divisor since $R$ is regular. Moreover, $R/x_iR$ has $(x_1, \ldots, x_i)$ generating $\bar{m}$. But $\dim R/x_iR = d-i$, thus $R/x_iR$ is regular local by definition. An induction finishes the proof. ($d-0$ case is trivial).

**Question:** $R$ regular local $\Rightarrow R_\mathfrak{p}$ regular local?

**Lemma.**

$R$: Noetherian local ring, $x \in m$. $M$ a finite module such that $x$ is a non-zero divisor on $M$ and $M/xM$ is free over $R/xR$. Then $M$ is free over $R$. ($x$ is necessarily a non-zero divisor in $R$)

Pf: $0 \to M \xrightarrow{x} M \to M/xM \to 0$ where $M/xM \cong (R/xR)^{\oplus r}$. Pick $m_1, \ldots, m_r$ which are mapped basis elements of $M/xM$. By Nakayama, $m_1, \ldots, m_r$ generate $M$. Thus we have a diagram: ($K = \ker \varphi$)

$$
\begin{array}{ccc}
R^{\oplus r} & \xrightarrow{x} & R^{\oplus r} \\
\downarrow \varphi & & \downarrow \varphi \\
0 & \to & M \xrightarrow{x} M \to M/xM \to 0 \\
\end{array}
$$

$\Rightarrow K \xrightarrow{x} K \to 0$ surjective $\Rightarrow K = 0$ by Nakayama and $\Rightarrow \varphi$ is an isomorphism.

**Lemma:**

If $R$ is a regular local ring, then any MCM is free.

Recall that if $R$ CM, then any module $M$ has a resolution:

$$0 \to K \to F_i \to \cdots \to F_0 \to M \to 0$$

(dimension shifting) with $K$ MCM. ($i \leq d-1$). Thus we have an immediate corollary:

**Cor.** If $R$ is regular local, then every finite $R$-module has a finite free resolution of length $\leq \dim R$. (More explicitly, we can find length $(\dim R \cdot \depth M)$ resolution.)

$\square$
Pf of lemma:
Assume M is MCM over R regular (CM).
d=0. R=k and the result is trivial.
Let xεm be part of a minimal system of generators of m. Then x is a
non-zero divisor on M (by the second lemma before, x is good w.r.t. the minimal
system, thus non-zero divisor). Moreover, R/xR is regular of dimension 1 less
and M/xM MCM over R/xR. Thus M/xM is free by induction and we are done
by the previous lemma.

Lemma: (Schanuel)
R: ring, M: an R-module. Suppose that 0→K↓→P_1→P_2→M→0,
0→L↓→P_2→M→0 are two s.e.s., and P_1, P_2 projective.
Then K⊕P_2≈L⊕P_1. (K, L are stably equivalent. If in particular, K is projective
then L is also injective).
Pf: Consider 0→ker(p+g)→P_1⊕P_2P→M→0

\[\begin{array}{ccc}
0 & \to & K \\
\uparrow & & \uparrow \\
0 & \to & P_1 \\
\uparrow & & \uparrow \\
0 & \to & M \\
\end{array}\]
Snake ⇒ ker(p+g)/K ≈ P_2. But P_2 projective ⇒ ker(p+g) ≈ K⊕P_2. By symmetry
K⊕P_2 ≈ ker(p+g) ≈ L⊕P_1. □

Def. R: Noetherian local. A finite R module M has finite projective dimension
if it has a finite length projective resolution, and projdim_R(M) ≡ minimal length
of such a resolution.

Lemma.
R: Noetherian local ring. M a finite R-module. Assume projdim_R(M)=d. If
F_e→F_{e-1}→⋯→F_0→M→0 is exact and F_i proj., e≥d-1. Then
ker(F_e→F_{e-1}) is finite projective.
□
Def. R: Noetherian. We say that R has finite global dimension if \( \exists \) an integer \( n \) s.t. every finite \( R \)-module has a finite projective resolution of length \( \leq n \). The minimal such \( n \) is called the global dimension of \( R \).

Cor. A regular local ring has finite global dimension \( \leq \dim R \).

Digression: What makes a complex exact?


Basic setting:

\((R, m, \kappa)\) Noetherian local ring.

Complex: \( 0 \rightarrow R^{n_e} \xrightarrow{\varphi_0} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \rightarrow R^{n_1} \xrightarrow{\varphi_1} R^{n_0} \rightarrow 0 \quad (\ast) \)

Lemma:

In the complex \((\ast)\), if any \( \varphi_i \) has a matrix coefficient a unit, then we can write \((\ast)\) up to isomorphism as a direct sum:

\[ 0 \rightarrow R^{n_e} \xrightarrow{\varphi_0} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \rightarrow R^{n_1} \xrightarrow{\varphi_1} R^{n_0} \rightarrow 0 \]

\[ \oplus \]

\[ 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow R \quad \rightarrow R \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \]

Pf: Simple linear algebra. \( \square \)

Complexes of the form \( 0 \rightarrow R \rightarrow R \rightarrow 0 \) are called trivial.

Cor. Every complex of the form \((\ast)\) is isomorphic to

\[ 0 \rightarrow R^{n_e} \xrightarrow{\varphi_0} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \rightarrow R^{n_1} \xrightarrow{\varphi_1} R^{n_0} \rightarrow 0 \]

\[ \oplus \]

Trivial ones.

and each \( \varphi_i \) has coefficients in \( m \).
Lemma:
If $R$ is Artinian, then any exact complex is a direct sum of trivial ones. 

Pf: In view of the previous cor., it suffices to show that $\ker \varphi_e \neq 0$, where $\varphi$ has its matrix coefficients in $m$.

But $R$ Artinian $\Rightarrow \exists x \in R, x \neq 0, m x = 0 \Rightarrow 0 \neq \left( \begin{array}{c} x \\ 0 \end{array} \right) \in \ker \varphi_e$. \hfill $\square$

Def: (Only for this digression) Given $\varphi : R^m \rightarrow R^n$, we set
1. $\text{rank} (\varphi) = \max \{ r \mid \Lambda^r \varphi : \Lambda^r R^m \rightarrow \Lambda^r R^n \text{ is non-zero} \}$
2. $I(\varphi) = \text{ideal generated by determinants of rank} \varphi \times \text{rank} \varphi \text{ minors of } \varphi$

Lemma: Suppose our complex

$$0 \rightarrow R^m \xrightarrow{\varphi} R^{n-e} \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi _{n-1}} R^2 \xrightarrow{\varphi_n} R^1 \xrightarrow{\varphi_n} 0$$

is a direct sum of trivial ones. Then:
1. $r_k(\varphi_i) \leq r_i = n_i - r_{i+1} + \cdots + (-1)^{e-i} n_e$
2. $r_k(\varphi_i) + r_k(\varphi_{i+1}) = n_i$
3. Each $I(\varphi_i) = R$.

Pf: Linear algebra. \hfill $\square$

Lemma: $\varphi : R^m \rightarrow R^n$, TFAE:
1. $\varphi$ is injective
2. rank($\varphi$) = $m$, and either $I(\varphi) = R$ or $I(\varphi)$ contains a non-zero divisor.

Pf: By the cor., we may assume $\varphi$ has all coefficients in $m$.

Let $q \in \text{Ass}(R), x \in R$ with $\text{ann}(x) = q$. Then

$$0 \rightarrow R^m \xrightarrow{\varphi} R^n$$

But $x R \subseteq R/q \Rightarrow (R/q)^m \xrightarrow{\varphi} (R/q)^n \Rightarrow (kq)^m \xrightarrow{\varphi} (kq)^n \Rightarrow \text{some } m \times m \text{ minor of } \varphi \text{ is not zero in } kq \Rightarrow I(\varphi) \subseteq q$. By prime avoidance, $\exists g \in I(\varphi), g \neq q, \forall q \in \text{Ass}(R) \Rightarrow g$ to a non-zero divisor.
Conversely, \( I(\phi) = R \Rightarrow \exists m \times m \) minor invertible, thus \( \phi \) must be injective. Furthermore, notice that \( \phi : R^m \to R^n \) is injective iff \( \phi_q : R_q^m \to R_q^n \) is injective \( \forall q \in \text{Ass} R \). (Indeed, \( \Rightarrow \) is easy; \( \Leftarrow \) \( \forall x \in R^m, \phi_q(x) = 0 \Rightarrow \phi_q(x) = 0 \Rightarrow x_q = 0 \Rightarrow \text{oannx} \subseteq q, \forall q \in \text{Ass} R \). By prime avoidance, \( \exists r \) a non-zero divisor, \( r \text{ann} x \Rightarrow x = 0 \).)

Now if \( x \in I(\phi) \) is a non-zero divisor, \( x \) becomes a unit in \( R_q, \forall q \in \text{Ass} (R) \) and it reduces to \( I(\phi) = R \) case. \( \square \)

Cor. If \( m \in \text{Ass} R \), then any exact complex is a direct sum of trivial ones. \( \square \)

Lemma. \( x \in R \) a non-zero divisor.

\[ 0 \to R^{ne} \overset{\phi_e}{\to} R^{ne-1} \overset{\phi_{e-1}}{\to} \cdots \to R^n \overset{\phi_n}{\to} R^{n-1} \overset{\phi_{n-1}}{\to} \cdots \to R^0 \overset{\phi_0}{\to} \to R \] exact

\[ \Rightarrow 0 \to (R/xR)^{ne} \to \cdots \to (R/xR)^{n_1} (\ast) \]

Pf: \( 0 \to (\ast) \to (\ast) \to (\ast) ' \to 0 \) to a s. e. s. of complexes

\[ \Rightarrow (l. e. s.) \ (\ast) ' \] is exact. \( \square \)

Lemma. (Acyclicity) \( R \) Noetherian local ring.

A complex \( M_* : 0 \to M_e \to M_{e-1} \to \cdots \to M_0 \) with depth \( M_i \geq i \).

Let \( i_0 \) be the largest index s.t. \( M_* \) is not exact at \( i \).

If \( i_0 > 0 \), then depth \( H_{i_0}(M_*) \geq 1 \).

Pf: We break the complex into parts:

\[ 0 \to M_e \to M_{e-1} \to I_{e-1} \to 0 \]

\[ 0 \to I_{e-1} \to M_{e-2} \to I_{e-2} \to 0 \]

\[ \vdots \]

\[ 0 \to I_{i_0+2} \to M_{i_0+1} \to I_{i_0+1} \to 0 \]

\[ 0 \to I_{i_0+1} \to \ker \phi_{i_0} \to H_{i_0} \to 0 \]

From a previous lemma, we know that

\[ \text{depth} I_{e-1} \geq \min \{ \text{depth} M_{e-1}, \text{depth} M_{e-1} \} \geq e-1 \]

\[ \text{depth} I_{e-2} \geq \min \{ \text{depth} M_{e-2}, \text{depth} I_{e-1} \} \geq e-2 \]

\[ \vdots \Rightarrow \text{depth} I_{i_0+1} \geq i_0+1, \quad \text{and} \quad \text{depth} H_{i_0} \geq \min \{ \text{depth} \ker \phi_{i_0}, \text{depth} I_{i_0+1} \} \geq i_0 > 0. \quad \square \]
Thm. \((R, m, x)\): Noetherian local ring. Consider the complex:
\[
0 \rightarrow R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \rightarrow R^n \rightarrow 0
\]
this complex is exact iff for all \(i, 1 \leq i \leq e\), we have:

1. \(\text{rank}(\varphi_i) = r_i\) (as predicted from trivial cases)
2. \(I(\varphi_i) = R_{\mathfrak{p}}\) or \(I(\varphi_i)\) contains an \(R_{\mathfrak{p}}\)-regular sequence of length \(i\).

Pf: W.L.O.G. we may assume that the matrix coefficients of all \(\varphi_i\) are in \(m\).
Firstly we show that \((1) + (2) \Rightarrow\text{exactness}\). For this purpose, we use induction on \(\dim R\).

\(\dim R = 0\), only \(I(\varphi_i) = R\) could occur, then some matrix coefficients must be units, contradiction with our assumption unless the original complex is a direct sum of trivial ones.

Next, observe we may assume \(e \geq 2\) by a previous lemma.
Claim: the complex localized at any prime \(\mathfrak{p} \subseteq R, \mathfrak{p} \neq m\), is exact.
This is true since \(\dim R_{\mathfrak{p}} < \dim R\) and conditions \((1)\) and \((2)\) still hold, thus the induction hypothesis applies.
It follows the cohomology groups \(H_*(M_{\mathfrak{p}})\) must be supported at \(m\) \(\Rightarrow \dim H_*(M_{\mathfrak{p}}) = 0\)
\((2)\) \(\Rightarrow \text{depth } R \geq e\). Thus by the acyclic lemma, if any \(H_*(M_{\mathfrak{p}}) \neq 0\), depth \(H_*(M_{\mathfrak{p}}) > 1\)
which contradicts the fact that \(\dim \geq \text{depth}\).

Conversely, assume that the complex is exact. \(\forall \varphi_i \in \text{Ass } R, \text{depth } R_{\varphi_i} = 0\) and the complex \(\otimes R_{\varphi_i}\) is exact. By a previous lemma, the localized complex must be a direct sum of trivial ones. Thus the ranks must then match \(I(\varphi_i) = R_{\varphi_i}\), \(\forall i\). Hence \(I(\varphi_i) \subseteq \varphi_i, \forall i\). \(\forall \varphi \in \text{Ass } R \Rightarrow \mathbf{I}(\varphi_i) \subseteq \varphi, \forall \varphi \Rightarrow \mathbf{I}(\varphi_i)\) contains a non-zero divisor in \(R\), by prime avoidance.

By a previous lemma, \(0 \rightarrow (R/x)^{n_e} \rightarrow (R/x)^{n_{e-1}} \rightarrow \cdots \rightarrow (R/x)^n\) is still exact.
By induction on \(e\) \((e = 1\) is a lemma\), \(I(\varphi_i)/xR \subseteq R/xR\) have depth \(\geq i-1\) and we are done. 

"How cool that is!"
Application:

Lemma.
Let \( R \) be a Noetherian local ring. Suppose the residue field \( k \) has finite projective dimension \( n \) over \( R \). Then \( \text{depth}_R k = n \).

\( R_k \): for many rings, \( \text{projdim}_R k = +\infty \).

\( \text{Pf} \): It suffices to prove for \( \text{projdim}_R k = n < \infty \), i.e., \( \exists \) an exact free complex

\[
0 \to P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \ldots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} k \to 0
\]

We may take the complex minimal so that the matrix coefficients of the boundary maps are in \( m \). Thm \( \Rightarrow I(q_i) \subseteq m \) contains a regular sequence of length \( n \).

\( \square \)

Lemma.
Suppose \( R \) is Noetherian local with maximal ideal \( m \), residue field \( k \). Then \( \text{Projdim}_R(k) \geq \dim_k m/m^2 \).

\( R_k \): note that we always have \( \dim R \leq \dim_k m/m^2 \), and for regular local rings \( \text{projdim}_R(k) \leq \dim R = \text{depth}_R \Rightarrow \dim R = \dim_k m/m^2 = \text{projdim}_R(k) \).

\( \text{Pf} \): Pick any finite free resolution of \( k \), with all matrix coefficients in \( m \).

\[ F_0 : 0 \to F_n \xrightarrow{e} F_{n-1} \xrightarrow{\ldots} F_1 \xrightarrow{\ldots} F_0 \xrightarrow{\ldots} k \to 0. \]

Pick \( x_1, \ldots, x_n \in m \) that form a basis of \( m/m^2 \). Form the Koszul complex,

\[ K_*(R, x_i) : 0 \to \wedge^n R^{n} \to \wedge^{n-1} R^{n} \to \ldots \to \wedge^2 R^n \to R^n \to R \]

with boundary maps:

\[ e_j : \wedge^{a-1} x_j \wedge e_j \rightarrow \frac{1}{a-1} \wedge^a e_j. \]

Clearly, \( H_0(K_*(R, x_i)) = R/(x_1, \ldots, x_n) = R/m = k \). Since \( F_0 \) is exact, and \( K_*(R, x_i) \) is a complex, a previous lemma \( \Rightarrow \exists \) a map of complexes inducing \( \text{id}_{H_0} = \text{id}_k \):

\[
\cdots \to \wedge^n R^n \to \wedge^{n-1} R^n \to \ldots \to R^n \to R \to k \to 0
\]

\[
\downarrow \alpha_{e} \downarrow \alpha_{e-1} \downarrow \alpha_{i} \downarrow \alpha_{0} \downarrow \text{id}
\]

\[
0 \to F_n \xrightarrow{e} F_{n-1} \xrightarrow{\ldots} F_1 \xrightarrow{\ldots} F_0 \xrightarrow{\ldots} k \to 0
\]

Observe that \( F_0 \) can be taken to be \( R \). Then \( \alpha_0 \) is necessarily multiplication by a unit.

\( \text{Claim} \): \( \alpha \circ k \) is injective.
Indeed: \[ R^n \otimes \mathfrak{m} \cong m \otimes \mathfrak{m} \]
\[ \downarrow \alpha_i \otimes \mathfrak{m} \downarrow \text{by a unit} \]
\[ F_i \otimes \mathfrak{m} \cong m \otimes \mathfrak{m} \]

\[ \Rightarrow \alpha_i \otimes \text{id}_\mathfrak{m} \text{ is injective} \]

Inductively, we shall show that \( \alpha_i \otimes \mathfrak{m} \) is injective. ∀ \( i \)

\[ \Lambda^i \mathfrak{m} \cong \Lambda^i R^n \otimes m \overset{\partial_i}{\longrightarrow} \Lambda^i(mR^n) \otimes \mathfrak{m} \cong \Lambda^i \Lambda^i R^n \otimes \mathfrak{m} \]
\[ \downarrow \alpha_i \otimes \mathfrak{m} \downarrow \alpha_i \otimes \mathfrak{m} \]
\[ F_i \otimes m \longrightarrow m F_i \otimes \mathfrak{m} \]

But by explicit computation, one can show that \( \partial_i \) is injective and by induction hypothesis \( \alpha_i \otimes \mathfrak{m} \) is injective ⇒ \( \alpha_i \otimes \mathfrak{m} \) is injective.

Hence we conclude that \( e \geq n \) and we are done. \( \square \)

Combining the above results, we obtain:

**Thm. (Characterization of Regular Local Rings)**

Let \( R \) be a Noetherian local ring. TFAE:

1. \( R \) is regular.
2. \( \mathfrak{m} \) has finite projective dimension.
3. \( R \) has finite global dimension.
4. For every finite \( R \)-module \( M \), we have \( \text{projdim}_R(M) + \text{depth}_R(M) = \text{dim} R \).

\( \text{Pf: } (4) \Rightarrow (3) \Rightarrow (2) \) is trivial.

(2) ⇒ (1) If \( \text{projdim}_R \mathfrak{m} < \infty \), \( \text{dim} R \geq \text{depth} R \geq \text{projdim} \mathfrak{m} \geq \text{dim} \mathfrak{m}/m^2 \geq \text{dim} R \).

(1) ⇒ (4) Recall that \( R \) regular local, of \( \text{dim} n \) (CM, in particular), \( M \) finite over \( R \) of depth \( e \), then \( M \) has a finite free resolution (dim shifting + MCM over regular local rings are free):

\[ 0 \rightarrow F_{n-e} \rightarrow \cdots \rightarrow F_i \rightarrow F_0 \rightarrow M \rightarrow 0 \]

Note that this is a minimal resolution since, again by dimension shifting, any minimal resolution has \( \text{dim} \text{ker} \varphi_i = n+i \). \( \square \)
Lemma:
Suppose $R$ has finite global dimension, then $S^IR$ has finite global dimension, \forall multiplicatively closed set.

Pf: Let $N$ be any finite $S^IR$ module, generated by $n_i, \ldots, n_n$ over $S^IR$. Let $M$ be the $R$-submodule generated by these $n_i$'s inside $N$. Then $M$ has a finite projective resolution: $0 \rightarrow P \rightarrow M \rightarrow 0$.

Localizing $(\otimes R S^IR) : 0 \rightarrow S^IR \rightarrow S^IM = N \rightarrow 0$ gives a projective resolution of $N$ as $S^IR$-module. □

Cor. $R$ regular local $\Rightarrow R_\mathfrak{p}$ regular local. □

Def. A Noetherian ring is called regular iff all of its local rings are regular.

⚠️ A regular local ring can have infinite dimension.

Cor. $R$ Noetherian. $R$ has finite global dimension $\iff$ $R$ is regular and $\dim R < \infty$.

Pf: $\Rightarrow$ by the lemma, $R_\mathfrak{p}$ has finite global dimension, \forall $\mathfrak{p}$. Thus $R_\mathfrak{p}$ regular.

$\Leftarrow$ \forall $R$-module $M$. Take a free resolution of length $\dim R - 1$:

$$0 \rightarrow K \rightarrow F_{d-1} \rightarrow \cdots \rightarrow F_i \rightarrow F_0 \rightarrow M \rightarrow 0$$

Now at every prime $\mathfrak{p}$, $K_\mathfrak{p}$ is free since $R_\mathfrak{p}$ regular with $\dim R_\mathfrak{p} \leq \dim R$. It follows that $K$ is projective. □
§12. Good Properties of Polynomial Rings

Later we shall study varieties (schemes).

Aff: the category of affine schemes (varieties) 
Alg: the category of finite algebras over $k$ (domains)

Via gluing, we will construct arbitrary schemes (varieties). Thus we first study some good properties of polynomial rings. (good properties of affine schemes, very primitive).

Dimension of finite type algebras over fields.

Lemma:
Let $m \subseteq k[x_1, \ldots, x_n]$ be a maximal ideal, then $m$ is generated by $n$ elements.

Pf: By Hilbert Nullstellensatz, $k(m)$ is a finite extension of $k$. We prove the result by induction. Let $m' = m \cap k[x_1, \ldots, x_{n-1}]$, then $k \subseteq k[x_1, \ldots, x_{n-1}] / m'$.

$\subseteq k(m) \Rightarrow m'$ is maximal. By induction hypothesis, $m' = (f_1(x_1, \ldots, x_{n-1}), \ldots, f_n(x_1, \ldots, x_{n-1}))$.

$\Rightarrow k(m) = k(m')[x_n] / (f_n(x_n))$ for some $f_n(x_n) \in k(m')[x_n]$, which can be taken to be a polynomial in $k[x_1, \ldots, x_n]$.

Lemma.
Assumption as above, then $\dim k[x_1, \ldots, x_n] / m = n$.

Pf: $0 \subseteq (f_1) \subseteq (f_1, f_2) \subseteq \cdots \subseteq (f_1, \ldots, f_n) = m$ is a chain of primes of length $n$. But $(f_1, \ldots, f_n)$ generates $m \Rightarrow \dim k[x_1, \ldots, x_n] / m = n$.

Cor. (1). $k[x_1, \ldots, x_n] / m$ is regular local of dim $n$.

(2). $k[x_1, \ldots, x_n]$ has finite global dimension $n$.

Note that in the previous section, to show (1) $\Rightarrow$ (2), we used that any locally free module is projective. By a conjecture (theorem now) of Serre states that any finite projective module over $k[x_1, \ldots, x_n]$ is actually free.
Properties:

1. Given a prime ideal \( \mathfrak{p} \subseteq k[x_1, \ldots, x_n] \), then \( k[x_1, \ldots, x_n]/\mathfrak{p} \) is equidimensional of dimension \( n - \text{ht}(\mathfrak{p}) \).

Proof: We did the proof before for the case \( k = \mathbb{K} \). Now, with the aid of the previous cor. we may remove the requirement \( k = \mathbb{K} \).

Suppose \( 0 = \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_r \) be a maximal chain of primes, let \( \mathfrak{p}_r \) be the preimages in \( k[x_1, \ldots, x_n] \), then \( \mathfrak{p}_r \) is necessarily maximal and \( \mathfrak{p}_0 = \mathfrak{p} \).

Furthermore, \( k[x_1, \ldots, x_n]/\mathfrak{p} \) is regular local \( \Rightarrow \) every maximal chain of primes descending from \( \mathfrak{p} \) has length \( \text{ht}(\mathfrak{p}) \). Combined, these chains give rise to a maximal chain of primes in \( k[x_1, \ldots, x_n]_m \) or \( k[x_1, \ldots, x_n]/(m) \Rightarrow \text{ht}(\mathfrak{p}) + \dim R/\mathfrak{p} = n. \)

2. \( \forall \mathfrak{p} \subseteq \mathfrak{q} \subseteq k[x_1, \ldots, x_n] \) primes. Then every maximal chain of primes between \( \mathfrak{p} \) and \( \mathfrak{q} \) has length \( \text{ht}(\mathfrak{q}) - \text{ht}(\mathfrak{p}) \).

Proof: \( k \) is u.c.

3. \( \dim k[x_1, \ldots, x_n]/\mathfrak{p} = \text{tr. deg}_k \text{ff.}(k[x]/\mathfrak{p}). \)

We shall prove the result using the next:

Lemma 1: \( k \) a field, \( (0) = \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_r = m \) a maximal chain of primes in \( k[x_1, \ldots, x_n] \). Then \( \exists k[y_1, \ldots, y_n] \hookrightarrow k[x_1, \ldots, x_n] \) finite s.t. \( \mathfrak{p}_i \cap k[y_1, \ldots, y_n] = (y_i, \ldots, y_n) \).

Proof of 3.

Say \( \mathfrak{p} = \mathfrak{p}_i \) in a maximal chain of primes as in Lemma 1. Then

\[ k[y_1, \ldots, y_n] = k[y_1, \ldots, y_n]/(y_i, \ldots, y_n) \hookrightarrow k[x]/\mathfrak{p} \]

and the map is finite by lemma 1.

\[ \Rightarrow k[y_1, \ldots, y_n] \hookrightarrow \text{ff.}(k[x]/\mathfrak{p}) \text{ is finite} \]

\[ \Rightarrow \text{tr. deg}_k k[x]/\mathfrak{p} = n - i = n - \text{ht}(\mathfrak{p}) = \dim k[x]/\mathfrak{p}. \]

Lemma 2. Given \( f \in k[x] \), non-constant, then \( \exists \varphi: k[y_1] \hookrightarrow k[x] \) finite, s.t. \( \varphi(y_1) = f \) and \( \varphi \) is an injection.
Pf: Write $f = \sum a_i x^{i}$ (some $a_i \neq 0$, $i \neq 0$, by assumption). W.L.O.G. we may assume that $x_i$ occurs in $f$. Consider

\[ f(x_i, y_1 + x_i^{e_1}, \ldots, y_n + x_i^{e_n}) = \sum_{i=0}^{\infty} \sum_{i_1=i, \ldots, i_n=i} (a_i x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + \text{l.o.t. in } x_i) \]

the l.o.t. has coefficients polynomials in $y_1, \ldots, y_n$. Now pick $1 < e_1 < e_2 < \cdots < e_n$ so that all integers $i_1 + i_2 + \cdots + i_n$ occurring are pairwise distinct. Hence:

\[ f = c \cdot x_i^{N} + \text{l.o.t. in } x_i, \quad c \in k. \]

Set $\varphi: y_1 \mapsto f, \quad y_2 \mapsto x_2 - x_i^{e_1}, \ldots, y_n \mapsto x_n - x_i^{e_n}.$

Claim: $x_i$ generates $k[x_i]$ over $k[y]$, as a ring, and $x_i$ satisfies a monic polynomial over $k[y]$.

Indeed, $\varphi(y_i) = f(x_i, \ldots, x_n) = f(x_1, \varphi(y_2) + x_i^{e_1}, \ldots, \varphi(y_n) + x_i^{e_n})$

\[ = c \cdot x_i^{N} + \text{l.o.t. in } x_i \]

$\Rightarrow c x_i^{N} + \cdots + c_n - \varphi(y_i) = 0$ is a monic equation.

That $\varphi$ is an injection is easy by def. of $\varphi$.

Rmk: A more general fact is that: $R \to S$, $S$ finitely generated by $x_1, \ldots, x_r$ over $R$, each $x_i$ satisfies a monic polynomial over $R$, then $S/R$ is finite.

Lemma 3. If $R \to S$ is a finite ring map, and $\mathfrak{p}, \mathfrak{p}'$ are distinct primes in $S$ lying over the same prime, then neither $\mathfrak{p}' \subset \mathfrak{p}$ nor $\mathfrak{p} \subset \mathfrak{p}'$.

Pf: Recall that primes lying over $\mathfrak{p} \leftrightarrow$ primes in $S \otimes_R k(\mathfrak{p})$. $S$ finite over $R \implies S \otimes_R k(\mathfrak{p})$ is finite over $k(\mathfrak{p})$, thus Artinian, and the distinct primes are all maximal, thus incomparable.

Proof of lemma 1

Pick $f \in \mathfrak{p}_i$, not a constant. Lemma 2 $\implies \exists k[y, z_1, \ldots, z_n] \subset k[x_i]$ finite such that $\varphi(y_i) = f \in \mathfrak{p}_i$. By lemma 3, all the prime ideals $(0) \cap k[y, z_1, \ldots, z_n] = k[y, z_1, \ldots, z_n] \cap \mathfrak{p}$, $\mathfrak{p}$ $\supseteq k[y, z_1, \ldots, z_n] \cap \mathfrak{p}$. Since $k[y, z_1, \ldots, z_n]$ is catenary, and has dim $n \implies k[y, z_1, \ldots, z_n] \cap \mathfrak{p} = (z_i)$. (Note that this also forces $(0) \cap k[y, z_1, \ldots, z_n] = (0)$ and $\varphi$ is injective).

Inductively, assume we have $k[y, z_1, \ldots, z_k, z_{k+1}] \subset k[x_1, \ldots, x_i]$ finite and $0 \subseteq \mathfrak{p} \subseteq k[y, z_1, \ldots, z_k, z_{k+1}] = (y_1, \ldots, y_i) \subseteq \mathfrak{p}$. $\mathfrak{p} \cap k[y, z_1, \ldots, z_k, z_{k+1}] = (z_i)$.
Choose any $f \in \mathbb{P}^m \cap \mathbb{K}[y,...,z; z_1, ..., z_n]$. If $f \in \mathbb{P}^m \cap \mathbb{K}[y,...,z; z_1, ..., z_n]$, we may write $f = g(z_1, ..., z_n) \cdot \frac{1}{h_i(y, z_1, ..., z_n)}$. Then $g \in \mathbb{P}^m \cap \mathbb{K}[y,...,z; z_1, ..., z_n] = (y_i, ..., y_i)$. Lemma 2 \Rightarrow \exists \text{ a finite map } \psi_0 : \mathbb{K}[y_1, z_1, ..., z_n] \rightarrow \mathbb{K}[z_1, ..., z_n] \text{ with } \psi_0(y_i) = g.

Take the base change, we obtain $\Psi : \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \rightarrow \mathbb{K}[z_1, ..., z_n] \text{ is still a finite map.}$ (Base change preserves finiteness.) Composing with the original map $\mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \rightarrow \mathbb{K}[z_1, ..., z_n]$, we obtain a finite map $\mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \rightarrow \mathbb{K}[z_1, ..., z_n]$. Moreover, $\mathbb{P}^m \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] = (y_1, ..., y_i) \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] = (y_i, ..., y_i)$ for $0 \leq k \leq i$ (which also forces the composite map to be injective again), and for $k = i+1$:

$\mathbb{P}^m \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] = (y_i, ..., y_i) \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \\
\geq (y_i, ..., y_i) \cap \mathbb{K}[y_i, ..., y_i, y_i, ..., z_i, z_i] \\
\geq (y_i, ..., y_i)$

Again by the cocatenary property of $\mathbb{K}[y_i, ..., y_i, z_i, z_i]$, we have $\mathbb{P}^m \cap \mathbb{K}[y_i, ..., y_i, z_i, z_i] = (y_i, ..., y_i)$.

and we are done. \square