Commutative Algebra

Note Title 9/4/2008

A Postgraduate course taught by
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Columbia University, Fall 2008
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- "Are there any undergraduate students?" Asked de Jong.

  One student in the back raised his hand

  "Generally I don't encourage undergrads to take algebraic geometry.

  You know, it could be, could be, you know..."

  He paused for a second

  "It's just like we should never talk about sex in front of kids!"

§1. The Spectrum of A Ring

Def. Let R be a ring

(1). The spectrum of R is the set of prime ideals of R, denoted $\operatorname{Spec} R$

(2). Given a subset $T \subset R$, we let $V(T)$ in $\operatorname{Spec} R$ be the set of prime ideals containing T : $V(T) \triangleq \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supset T \}$

(3). Given an element $f \in R$, we let $D(f) \subset \operatorname{Spec} R$ be the set of primes $\mathfrak{p}$ not containing f.

Lemma:

(1). $\operatorname{Spec} R$ is empty iff $R = (0)$

(2). Every nonzero ring has a maximal ideal

(3). Every nonzero ring has a minimal prime ideal.

(4). Given $I \subseteq R$ and a prime ideal $\mathfrak{p} \supseteq I$, then there exists a prime ideal $\mathfrak{q}$ $I \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ with $\mathfrak{q}$ minimal over I.

(5). If $T \subseteq R$, then $V(I) = V((T))$

(6). Given $I \subseteq R$, $\sqrt{I} \triangleq \{ f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N} \} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$

(7). If I is an ideal, then $V(I) = V(\sqrt{I})$

(8). If I is an ideal, and $V(I) = \phi \Leftrightarrow I = R$; $V(0) = \operatorname{Spec} R$.

(9). I, J ideals, then $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

(10). If $(I_i)_{i \in A}$ is a collection of ideals, $\bigcap_{i \in A} V(I_i) = V(\sum_{i \in A} I_i)$

(11). If $f \in R$, $\operatorname{Spec} R = V(f) \amalg D(f)$

(12). If $f \in R$, then $D(f) = \phi \Leftrightarrow f$ is nilpotent.

(13). If $f, f' \in R$, $f = uf'$ with u a unit $\Rightarrow D(f) = D(f')$

(14). If I is an ideal, $\mathfrak{p}$ is a prime, $\mathfrak{p} \notin V(I) \Rightarrow \exists f \in I$ s.t. $\mathfrak{p} \in D(f)$ and $D(f) \cap V(I) = \phi$.

(15). $f, g \in R \Rightarrow D(f) \cap D(g) = D(fg)$

Def. Let R be a ring. The topology on $\operatorname{Spec} R$ whose closed sets are $V(T)$ is called the Zariski topology ((8), (9), (10) of lemma $\cdots$). The open sets $D(f)$ are called standard open. They form a basis of the Zariski topology.

Proof of lemma:

(1) "$\Leftarrow$" easy.

"$\Rightarrow$" $R \neq (0) \Rightarrow 1 \neq 0$ in R, by (2), Spec$R$ is not empty.

(2) R non-zero $\Rightarrow 1 \neq 0$ in R. Let $\mathcal{S} = \{$ideals I in $R\}$ (R is not considered as an ideal) As a partially ordered set where "$\leq$" is defined by $I \leq I' \Leftrightarrow I \subseteq I'$, $\mathcal{S}$ is not empty since $(0) \in \mathcal{S}$. Thus if we can show that if any totally ordered subset of $\mathcal{S}$ has an upper bound, then $\mathcal{S}$ will have a maximal element by Zorn's lemma, which is nothing but a maximal ideal. Indeed, let $\{I_\alpha\}_{\alpha \in A}$ be a totally ordered subset of $\mathcal{S}$.

Claim: $\bigcup_{\alpha \in A} I_\alpha \in \mathcal{S}$.

Clearly $1 \notin \bigcup_{\alpha \in A} I_\alpha$. Also if $x, y \in \bigcup_{\alpha \in A} I_\alpha \Rightarrow x \in I_\alpha$, $y \in I_{\alpha'}$ for $\alpha, \alpha' \in A$. Since A is totally ordered, W.L.O.G. $\alpha \leq \alpha' \Rightarrow I_\alpha \subseteq I_{\alpha'} \Rightarrow x, y \in I_{\alpha'} \Rightarrow x+y \in I_{\alpha'} \subseteq \bigcup_{\alpha \in A} I_\alpha$. Moreover $ax \in I_\alpha \subseteq \bigcup_{\alpha \in A} I_\alpha$. Thus $\bigcup_{\alpha \in A} I_\alpha$ is an ideal and is an upper bound for $\mathcal{S}$.

(3). Similar proof as in (2), with the partial relation changed into "$I \leq I' \Leftrightarrow I \supseteq I'$" and $\mathcal{S}$ is non-empty since we have maximal ideals by (2).

"Let me tell you something about sets"

Remarked de Jong. Everybody concentrated.

"It's very big~~"

(4). Apply (3) to the ring A/I

(5). Any $\mathfrak{p} \supseteq T$ certainly contains (T)

(6). Let $\mathfrak{p}$ be in $V(I)$. If $x \in \sqrt{I} \Rightarrow x^n \in I \Rightarrow x^n \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \Rightarrow \sqrt{I} \subseteq \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$

Conversely, $x \notin \sqrt{I} \Rightarrow \forall n \in \mathbb{N}$, $x^n \notin I$. Let $S = \{1, x, x^2, \cdots\}$, then $S^{-1}R = R_x$ is non-zero since $0 \notin S \Rightarrow \exists \mathfrak{p}'$ of R_x containing IR_x by (2). The preimage of $\mathfrak{p}'$ in R is a prime ideal containing I and not meeting $S \Rightarrow x \notin \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$

(7). $I \subseteq \sqrt{I} \Rightarrow V(I) \supseteq V(\sqrt{I})$. Conversely $\mathfrak{p} \supseteq I \Rightarrow \mathfrak{p} \supseteq \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p} = \sqrt{I} \Rightarrow \mathfrak{p} \in V(\sqrt{I})$

(8). $\mathfrak{p} \supseteq I \Leftrightarrow \bar{\mathfrak{p}} \supseteq 0$ in R/I. The result follows from (1) then

(9). $I(J) \supseteq I \cap J \supseteq IJ \Rightarrow V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$. Conversely, if $\mathfrak{p} \supseteq IJ$ but not $I \Rightarrow \exists x \in I$, $x \notin \mathfrak{p}$. Moreover, $\forall y \in J$, $xy \in \mathfrak{p} \Rightarrow y \in \mathfrak{p} \Rightarrow \mathfrak{p} \supseteq J$.

(10). $I_\alpha \subseteq \sum I_\alpha \Rightarrow V(I_\alpha) \supseteq V(\sum I_\alpha) \Rightarrow \bigcap_{\alpha \in A} V(I_\alpha) \supseteq V(\sum I_\alpha)$. Conversely $\mathfrak{p} \in \bigcap_{\alpha \in A} V(I_\alpha)$ $\Rightarrow \mathfrak{p} \supseteq I_\alpha$, $\forall \alpha \in I_\alpha \Rightarrow \mathfrak{p} \supseteq \sum_{\alpha \in A} I_\alpha$.

(11). It's just a definition.

(12). $D(f)=\phi \Leftrightarrow V(f)=\mathrm{Spec}R \Leftrightarrow f\in\bigcap_{\text{all }\mathfrak{p}}\mathfrak{p}=\sqrt{0} \Leftrightarrow f$ nilpotent

(13). $f=uf' \Rightarrow (f)=(f') \Rightarrow V(f)=V((f))=V((f'))=V(f') \Rightarrow D(f)=D(f')$

(14). $\mathfrak{p}\notin V(I) \Rightarrow \mathfrak{p}\not\supseteq I \Rightarrow \exists f\in I$ s.t. $f\notin\mathfrak{p} \Rightarrow \mathfrak{p}\in D(f)$ and $\forall\ q\in D(f),\ q\notin V(I)$.

(15). Take complement of (9) where $I=(f)$ $J=(g)$. □

Examples :

1). Spec $\mathbb{Z}$ $=\{(2),(3),(5),\cdots,(0)\}$

(0) is not closed. since $(0)=V(I)\Rightarrow (0)\supseteq I \Rightarrow I=(0)$. But $V(0)=\mathrm{Spec}\,\mathbb{Z}$.

All other primes are closed.

2). Spec $(\mathbb{Z}[x]/(x^2-4))$

We have $0\to\mathbb{Z}\to\mathbb{Z}[x]/(x^2-4) \Rightarrow \mathrm{Spec}\,\mathbb{Z}[x]/(x^2-4)\to\mathrm{Spec}\,\mathbb{Z}$, $\mathfrak{p}\mapsto\mathfrak{p}\cap\mathbb{Z}$

Case I : $\mathfrak{p}\cap\mathbb{Z}=(2)$.

$\Rightarrow\ 0\to\mathbb{Z}/2\to\mathbb{Z}/2[x]/(x^2-4)=\mathbb{Z}/2[x]/(x^2)$

The only prime in $\mathbb{Z}/2[x]/(x^2)$ is (x) of multiplicity 2.

$\Rightarrow$ The only $\mathfrak{p}\cap\mathbb{Z}=(2)$ is $(2,x)$

Case II. $\mathfrak{p}\cap\mathbb{Z}=(p)$, p odd.

$\Rightarrow\ 0\to\mathbb{Z}/p\to\mathbb{Z}/p[x]/(x^2-2^2)=\mathbb{Z}/p[x]/((x-2)(x+2))$

The only primes in $\mathbb{Z}/p[x]/(x^2-2^2)$ are $(x-2)$ and $(x+2)$

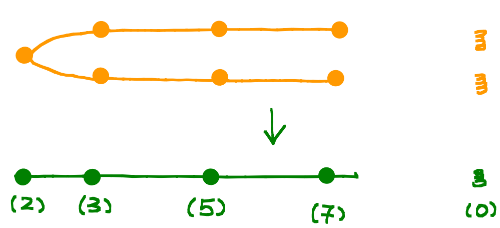
$\Rightarrow$ The only $\mathfrak{p}\cap\mathbb{Z}=(p)$ are $(p,x-2)$, $(p,x+2)$

Case III. $\mathfrak{p}\cap\mathbb{Z}=(0)$

$\Rightarrow\ 0\to\mathbb{Q}\to\mathbb{Q}[x]/(x^2-4)$

The prime ideals are $(x-2)$ and $(x+2)$ in $\mathbb{Q}[x]/(x^2-4)$.

$\Rightarrow$ The only primes $\mathfrak{p}\cap\mathbb{Z}=(0)$ are $(x-2)$, $(x+2)$.



3). Spec $\mathbb{Z}[x]$

Similar as above. consider $0\to\mathbb{Z}\to\mathbb{Z}[x]$, let $\mathfrak{p}$ be in Spec $\mathbb{Z}[x]$

Case I $\mathfrak{p} \cap \mathbb{Z} = (p)$

$\Rightarrow 0 \to \mathbb{Z}/p \to \mathbb{Z}/p[x]$

Since $\mathbb{Z}/p[x]$ is a P.I.D. $\Rightarrow$ All primes are of the form $(\bar{f})$ where $\bar{f}$ is an irreducible polinomial of $\mathbb{Z}/p[x]$ or (0).

$\Rightarrow \mathfrak{p} = (p, f)$ where f is a preimage of $\bar{f}$ under modulo p or $\mathfrak{p} = (p)$

Case II. $\mathfrak{p} \cap \mathbb{Z} = (0)$

$\Rightarrow 0 \to \mathbb{Q} \to \mathbb{Q}[x]$

Since $\mathbb{Q}[x]$ is a P.I.D. $\Rightarrow$ All primes are of the form (f) where f is a primitive irreducible polynomial in $\mathbb{Z}[x]$ or (0)

$\Rightarrow \mathfrak{p} = (f)$

4) Spec $k[x, y]$

This is similar as the case of Spec $\mathbb{Z}[x]$, since $k[x]$, similar as $\mathbb{Z}$, is a P.I.D.

Spec $k[x] = \{(f) \mid f$ an irreducible polynomial$\} \cup \{(0)\}$, and (0) is not closed.

Thus similarly as in 3), consider $0 \to k[x] \to k[x,y]$, and let $\mathfrak{p} \in \operatorname{Spec} k[x,y]$

Case I. $\mathfrak{p} \cap k[x] = (f)$

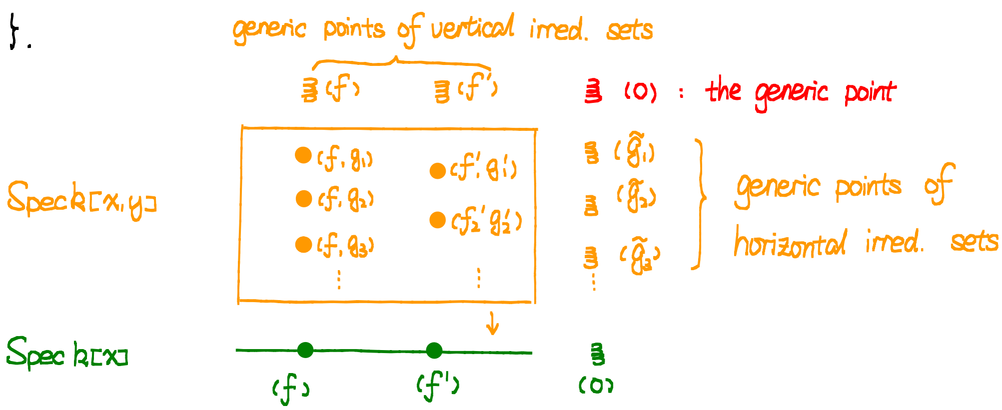
$\Rightarrow 0 \to \frac{k[x]}{(f)} \to \frac{k[x]}{(f)}[y]$

Since $\frac{k[x]}{(f)}$ is a field, $\frac{k[x]}{(f)}[y]$ is a P.I.D. with ideals of the form $(\overline{g(x,y)})$ where $g(x,y)$ is a polynomial which mod (f) is irreducible in $\frac{k[x]}{(f)}[y]$, and the ideal (0) of $\frac{k[x]}{(f)}[y]$. Thus such $\mathfrak{p}$'s are $\{(f(x), g(x,y)), \cdots, (f(x))\}$

Case II. $\mathfrak{p} \cap k[x] = (0)$

$\Rightarrow 0 \to k(x) \to k(x)[y]$

$k(x)[y]$ is a P.I.D. $\Rightarrow$ all primes are of the form $(\tilde{g}(x,y))$ where $\tilde{g}(x,y)$ is an irreducible polynomial in $k[x,y]$, and the ideal (0). Thus such $\mathfrak{p}$'s are $\{(\tilde{g}(x,y)), \cdots, (0)\}$.



Lemma:

If $\varphi: R \longrightarrow S$ is a ring map, then $\operatorname{Spec} S \xrightarrow{\operatorname{Spec}(\varphi)} \operatorname{Spec} R$, $\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$ is continuous and in fact the preimage of standard open sets are standard open sets, i.e.

$$\operatorname{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f)) \ . \quad f \in R .$$

Moreover, if $R \xrightarrow{\varphi} R' \xrightarrow{\varphi'} R'' \Rightarrow \operatorname{Spec} R'' \xrightarrow{\operatorname{Spec}\varphi'} \operatorname{Spec} R' \xrightarrow{\operatorname{Spec}\varphi} \operatorname{Spec} R$, and

$$\operatorname{Spec}\varphi \circ \operatorname{Spec}\varphi' = \operatorname{Spec}(\varphi' \circ \varphi)$$

i.e. Spec is a contravariant functor. $\square$

Lemma:

$R \longrightarrow S^{-1}R \Rightarrow \operatorname{Spec}(S^{-1}R) \longrightarrow \operatorname{Spec} R$. This is a *homeomorphism onto the image*, where the image $= \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \phi\}$.

Pf: By basic properties of localization, $\operatorname{Spec}(S^{-1}R)$ is in 1-1 correspondence with primes in R not meeting S. Moreover, by the previous lemma, the map is continuous. Thus it suffices to check that the map is open. Let $D(f)$ be an open set in $\operatorname{Spec}(S^{-1}R)$, then $f = \frac{h}{s}$, $h \in R$, $s \in S$. $f \in S^{-1}\mathfrak{p} \Leftrightarrow \frac{h}{s} = \frac{p}{s'}$ for some $p \in \mathfrak{p} \Leftrightarrow s''(s'h - ps) = 0$ for some $p \in \mathfrak{p}$ $\Leftrightarrow sh \in \mathfrak{p}$ for some $s \in S \Leftrightarrow h \in \mathfrak{p}$ (since $\mathfrak{p} \cap S = \phi$). Thus the image of $D(f)$ is just $D(h)$ and the map is open. $\square$

- A special case of the lemma: $S = \{1, f, f^2, \cdots\}$. Then

$$\begin{tikzcd} \operatorname{Spec}(R_f) \triangleq \operatorname{Spec}(S^{-1}R) \arrow[r] \arrow[dr, "\sim"] & \operatorname{Spec} R \arrow[d, phantom, "\cup\!\mathsf{I}"] \\ & D(f) \end{tikzcd}$$

Lemma:

Let $I \subseteq R$ be an ideal. $\operatorname{Spec}(R/I) \longrightarrow \operatorname{Spec}(R)$ is a homeomorphism onto its image $V(I)$.

Pf: Similar as above it suffices to check that the map is open. Let $D(\bar{f})$ be an open set of $\operatorname{Spec}(R/I)$. $\bar{\mathfrak{p}} \supseteq (\bar{f}) \Leftrightarrow \mathfrak{p} \supseteq I$ and $\mathfrak{p} \supseteq (f) \Leftrightarrow \mathfrak{p} \supseteq I + (f) \Leftrightarrow$ $\mathfrak{p} \in V(I) \cap V(f)$. $\square$

• Recall some general facts about localization: R is a ring, S: a multiplicatively closed subset $S^{-1}R \triangleq \{\frac{r}{s} \mid r \in R,\ s \in S\}/\sim$, where $\frac{r}{s} \sim \frac{r'}{s'} \Leftrightarrow \exists\, s'' \in S$ s.t. $s''(rs'-r's)=0$.

Examples:

1). $S=\{1, f, f^2, f^3 \cdots\}$ for some $f \in R$

In this case $\operatorname{Spec} S^{-1}R \cong D(f) \subseteq \operatorname{Spec} R$ as a canonical open set.

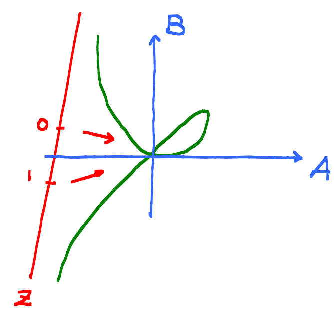
e.g. $R=\mathbb{C}[x]$, $f=x(x-1)$ $R_f=\mathbb{C}[x, \frac{1}{x(x-1)}]$

$\operatorname{Spec} \mathbb{C}[x, \frac{1}{x(x-1)}] \cong D(f) = \operatorname{Spec} \mathbb{C}[x] \setminus \{(x), (x-1)\}$

2). $R=\{f \in \mathbb{Q}[z] \mid f(0)=f(1)\} \subseteq \mathbb{Q}[z]$

Claim: $R \cong \mathbb{Q}[A, B]/(A^3-B^2+AB)$ where $A \mapsto z^2-z$, $B \mapsto z^3-z^2$

It's easy to check that $\mathbb{Q}[A, B] \longrightarrow R$ is well-defined, and since the image is inside an integral domain, the kernel is a prime ideal $\mathfrak{P}$ of $\mathbb{Q}[A, B]$ and certainly contains the element A^3-B^2+AB. From example 4) above, we know that $\operatorname{Spec}\mathbb{Q}[A, B]$ consists of four types of elements: $\{(f(A)), (f(A), g(A,B)), (g(A,B)), (0)\}$. But $\dim_{\mathbb{Q}} \mathbb{Q}[A,B]/(f(A), g(A,B)) < +\infty$ but $\dim_{\mathbb{Q}} R = +\infty$



$\Rightarrow \mathfrak{P}$ is of the form $(g(A,B))$, and since A^3-B^2+AB is irreducible, $\mathfrak{P}=(A^3-B^2+AB)$.

Now pick $a \in \mathbb{Q}$ with $a \neq 0, 1, \frac{1}{2}$, and let $R_a=\{f \in \mathbb{Q}[z, \frac{1}{z-a}] \mid f(0)=f(1)\}$ $\subseteq \mathbb{Q}[z, \frac{1}{z-a}]$

• Claim: R_a is a finitely generated $\mathbb{Q}$-algebra

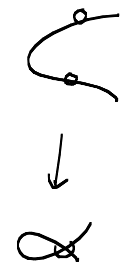
• Claim: $\operatorname{Spec} R_a$ maps homeomorphically onto an open subset of $\operatorname{Spec} R$ $\operatorname{Spec} R \setminus \{\text{a closed point}\}$, but R_a is not isomorphic to $S^{-1}R$ for any multiplicatively closed subset of R

$$\{f \in R[z] \mid f(0) = f(1)\}$$
$$\|$$
$$\frac{\mathbb{Q}[A,B]}{(A^3 - B^2 + AB)} \xrightarrow{\cong} R \hookrightarrow R[z]$$

$A \longmapsto z^2 - z$

$B \longmapsto z^3 - z^2$

$a + b + c = c$ $\qquad g = \Sigma$



$f(0) = f(1) = c$

$g(x)$ $\quad f(x) - c$

$g(0) = 0 \quad g(1) = 0$

$z(z-1) \mid g(0)$

smooth

$\mathbb{Q}[A,B]$

$A = A$

$B = Au$

$u - u^2 + A$

$$A^3 - A^2u^2 + A^2 u$$
$$= A^2(u - u^2 + A)$$

Def. A local ring is a ring with only 1 maximal ideal

Lemma:

Let R be a ring. then TFAE:

(1). R is local

(2). R has a maximal ideal $\mathfrak{m}$ and all elements in $R\setminus\mathfrak{m}$ are units.

Pf: (1) $\Rightarrow$ (2). Every non-unit is contained in a maximal ideal, thus in $\mathfrak{m}$

(2) $\Rightarrow$ (1). If $\mathfrak{n}$ is an ideal, $\mathfrak{n}$ consists of non-units $\Rightarrow \mathfrak{n}\subseteq\mathfrak{m}$. $\square$

Notation: R a ring. $\mathfrak{p}$ a prime ideal, $R_\mathfrak{p} \triangleq (R\setminus\mathfrak{p})^{-1}R$, called the localization of R at $\mathfrak{p}$.

Fact: $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Note that $\operatorname{Spec} R_\mathfrak{p} = \{\mathfrak{q}\in\operatorname{Spec}R \mid \mathfrak{q}\cap(R\setminus\mathfrak{p}) = \phi\} = \{\mathfrak{q}\in\operatorname{Spec}R \mid \mathfrak{q}\subseteq\mathfrak{p}\}$.

Def. The residue field of $\mathfrak{p}$ is the field $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ = Fraction field of $R/\mathfrak{p}$.

Fundamental diagram: $R\xrightarrow{\varphi} S$ a ring map, and $\mathfrak{p}\subseteq R$ a prime ideal.

$$\begin{tikzcd}
 & S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} & S_\mathfrak{p} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{p}S \arrow[r] & (R/\mathfrak{p}\setminus 0)^{-1}S/\mathfrak{p}S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \\
\kappa(\mathfrak{p}) = & R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \arrow[u] & R_\mathfrak{p} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u, "\varphi"'] & R/\mathfrak{p} \arrow[r] \arrow[u] & (R/\mathfrak{p}\setminus 0)^{-1}R/\mathfrak{p} = \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

Any of the squares induce geometrically a fiber product. The upshot of this diagram is that $\mathfrak{p}$ is in the image of $\operatorname{Spec}(\varphi): \operatorname{Spec}(S)\longrightarrow\operatorname{Spec}R$ iff $\kappa(\mathfrak{p})\otimes_R S\neq 0$ or equivalently $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$ or $\mathfrak{p}S_\mathfrak{p}\neq S_\mathfrak{p}$.

Lemma:

$\operatorname{Spec}R$ is quasi-compact.

Pf: Since $\{D(f) \mid f\in R\}$ forms a basis of the Zariski topology, it suffices to show that, if $\operatorname{Spec}R = \bigcup_{i\in I} D(f_i)$, then $\exists$ a finite subset T of I s.t. $\operatorname{Spec}R = \bigcup_{i\in T} D(f_i)$.

Indeed, $\operatorname{Spec}R = \bigcup_{i\in I} D(f_i) \Leftrightarrow \phi = \bigcap_{i\in I} V(f_i) \Leftrightarrow \phi = V(\sum_{i\in I}(f_i)) \Leftrightarrow \sum_{i\in I}(f_i) = R \Leftrightarrow \exists\, T$, a finite subset of I s.t $1 = \sum_{i\in T} a_i f_i \Leftrightarrow \phi = V(\sum_{i\in T}(f_i)) \Leftrightarrow \operatorname{Spec}R = \bigcup_{i\in T} D(f_i)$ $\square$

Example:

$R = \prod_{n\in\mathbb{N}} \mathbb{F}_2 \xrightarrow{p_i} \mathbb{F}_2 \Rightarrow \ker p_i$ is a maximal ideal (prime). Note that $\forall x \in R$, $x^2 = x \Rightarrow \forall \mathfrak{p}$ prime. $R/\mathfrak{p} \cong \mathbb{F}_2$. But are $\{\ker p_i\}$ all the prime ideals in R?

If yes $\Rightarrow \operatorname{Spec} R = \{\ker p_i\}$ is a countable set with discrete topology, which cannot be quasicompact, contradiction with the lemma. $\Rightarrow \exists$ "funny" primes. One such prime ideal can be seen as follows: $\bigoplus \mathbb{F}_2 \hookrightarrow \prod \mathbb{F}_2$ as an ideal, which must be contained in a maximal ideal, any such a maximal ideal is not a $\ker p_i$.

Def. Let X be a topological space

(1) X is connected iff ($X = T_1 \sqcup T_2$, T_i both open and closed $\Rightarrow T_1$ or T_2 is $\emptyset$).

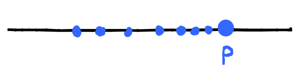
(2) We say T is a connected component of X if it is a maximal connected subset.

Lemma.

X a topological space, $T \subseteq X$ is connected $\Rightarrow \overline{T}$ is connected. Every $x \in X$ is contained in a connected component. Connected components are always closed but not necessarily open. $\square$

Example: a closed connected component which is not open

A sequence of points converge to a point p, then $\{p\}$ is a connected component but not open.



Def: An idempotent of a ring is an element $e \in R$ satisfying $e^2 = e$. Trivial idempotents are $1, 0$, (which are the only ones in a domain).

If e is an idempotent, so is $1-e$. The standard open set $D(e) = \{\mathfrak{p} \in \operatorname{Spec} R \mid e \notin \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \bar{e} \neq 0 \text{ in } R/\mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \bar{e} = 1 \in R/\mathfrak{p} \subseteq \kappa(\mathfrak{p})\}$
$= \{\mathfrak{p} \in \operatorname{Spec} R \mid e \mapsto 1 \in \kappa(\mathfrak{p})\}$

Similarly, $D(1-e) = \{\mathfrak{p} \in \operatorname{Spec} R \mid e \mapsto 0 \in \kappa(\mathfrak{p})\}$.

Since 1, 0 are the only possible images in $k(\mathfrak{p})$ for our idempotents, we conclude that $\operatorname{Spec} R = D(e) \amalg D(1-e)$.

Lemma:

The construction $e \mapsto D(e)$ gives a bijection between:

$\{\text{idempotents of } R\} \longleftrightarrow$ Open and closed subsets of $\operatorname{Spec} R$.

$$e \leftrightarrow D(e)$$

Upshot: $\operatorname{Spec} R$ is connected iff R has only trivial idempotents

To prove the lemma, we first show:

Lemma:

Given a ring R and elements $f_1, \cdots, f_r$ s.t. $\langle f_1, \cdots, f_r \rangle = R$, then the sequence

$$0 \longrightarrow R \xrightarrow{\alpha} \bigoplus_i R_{f_i} \xrightarrow{\beta} \bigoplus_{i,j} R_{f_i f_j}$$

is exact, where $\alpha(f) = (\frac{f}{1}, \cdots, \frac{f}{1})$ $\beta(x_1, \cdots, x_n) = (x_i - x_j)_{i,j}$

Pf: $\alpha(f) = 0 \Rightarrow f_i^{k_i} f = 0$ $i = 1, \cdots, r$, and we may take a common k s.t. $f_i^k f = 0$

But $(f_1^k, \cdots, f_r^k) = R \Rightarrow 1 = \sum a_i f_i^k \Rightarrow f = \sum a_i f_i^k f = 0$

Obviously, $\beta \circ \alpha = 0$. Moreover, if $\beta(\frac{a_1}{f_1^{k_1}}, \cdots, \frac{a_r}{f_r^{k_r}}) = 0$, i.e. $(a_i f_j^{k_j} - a_j f_i^{k_i})(f_i f_j)^{N_{ij}} = 0$ $\forall i, j$. Since there are only finitely many terms involved, we may take a common N s.t. $(a_i f_j^{k_j} - a_j f_i^{k_i})(f_i f_j)^N = 0 \Rightarrow a_i f_j^{k_j+N} f_i^N = a_j f_i^{k_i+N} f_j^N$. Replace a_i by $a_i f_i^N$ f_i by $f_i^{k_i+N}$, we may assume the element is given by $(\frac{a_1}{f_1}, \cdots, \frac{a_r}{f_r})$ and $f_i a_j = f_j a_i$

Furthermore $\langle f_1, \cdots, f_r \rangle = 1 \Rightarrow \exists b_j$ s.t. $\sum b_j f_j = 1$. Now take $g = \sum b_j a_j \in R$. then $f_i g = \sum b_j a_j f_i = \sum b_j a_i f_j = (\sum b_j f_j) a_i = a_i \Rightarrow g \mapsto \frac{a_i}{f_i}$ in R_{f_i}. $\square$

Proof of the lemma.

If V is an open and closed set, so is V^c, thus we may cover each of them by finitely many standard open sets, say $V = \bigcup_{i=1}^{r} D(f_i)$, $V^c = \bigcup_{j=1}^{s} D(h_j)$. Since $V \cup V^c = \operatorname{Spec} R \Rightarrow \langle f_1, \cdots, f_r, h_1, \cdots, h_s \rangle = R$.

Consider $0 \to R \xrightarrow{\alpha} \oplus R_{f_i} \oplus \oplus R_{h_j} \xrightarrow{\beta} \oplus R_{f_i f_j} \oplus R_{h_j h_k}$ (note $R_{f_i h_j} = 0$)

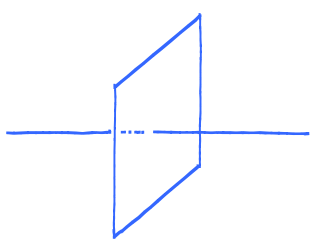
Let $\tilde{e}$ be the element $(\overbrace{1, \cdots, 1}^{r}, \overbrace{0, \cdots, 0}^{s}) \in \oplus R_{f_i} \oplus \oplus R_{h_j}$

Then $\beta(\tilde{e}) = 0$, thus by the previous lemma $\tilde{e} = \alpha(e)$ for a unique e, and e maps to 1 in every $\kappa(\beta)$ for $\beta \in \bigcup_{i=1}^{r} D(f_i)$ and 0 in $\kappa(\beta)$ for every $\beta \in \bigcup_{j=1}^{s} D(h_j)$.
Moreover $\alpha(e^2) = \tilde{e}^2 = \tilde{e} = \alpha(e)$ and by injectivity of α, $e^2 = e$, thus e is an idempotent.
The uniqueness of e also follows from the injectivity of α. $\square$

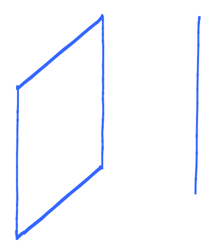
Def: Let X be a topological space.
(1). We say X is irreducible if ($X = Z_1 \cup Z_2$, Z_i closed in $X \Rightarrow Z_1 = X$ or $Z_2 = X$).
(2). An irreducible component of X is a maximal irreducible subset of X.

Lemma:
Let X be a topological space. $T \subseteq X$ irreducible $\Rightarrow \overline{T}$ is irreducible. Irreducible components are closed and $\forall x \in X$ is contained in an irreducible component of X.
$\square$

Examples:
(1) Connected components and irreducible components.



One connected component
Two irreducible components



Two connected components
Two irreducible components.

(2). A singleton set is irreducible $\Rightarrow x \in X$, $\overline{\{x\}}$ is irreducible.

Def. Let X be a topological space.
(1). Let $Z \subseteq X$ be closed, irreducible. A generic point of Z is a point $\xi \in X$ s.t. $\overline{\{\xi\}} = Z$
(2). The space X is called sober if every irreducible closed subset has a unique generic point.

Example:

Suppose R is a domain. Then $(0) \in \operatorname{Spec} R$ and $\overline{\{(0)\}} = \operatorname{Spec} R$.
It follows that $\operatorname{Spec} R$ is irreducible and (0) is the generic point.

Lemma:

Let R be a ring, the irreducible closed subsets of $\operatorname{Spec} R$ are exactly the subsets $V(\mathfrak{p})$ with $\mathfrak{p}$ prime. The irreducible components of $\operatorname{Spec} R$ are the subsets $V(\mathfrak{p})$ with $\mathfrak{p}$ minimal prime

Upshot: $\operatorname{Spec} R$ is a sober topological space.

Recall that $V(I) = V(\sqrt{I})$, $\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$ $\Rightarrow \exists$ 1:1 correspondence between

$$\begin{array}{ccc} \{\text{closed subsets of } \operatorname{Spec} R\} & \longleftrightarrow & \{\text{radical ideals of } R\} \\ Z & \longmapsto & \bigcap_{\mathfrak{p} \in Z} \mathfrak{p} \\ V(I) & \longleftarrow\!\shortmid & I \end{array}$$

Proof of lemma: It suffices to consider radical ideals. Let I be radical. $V(I)$ irreducible $\overset{?}{\Leftrightarrow}$ I is a prime

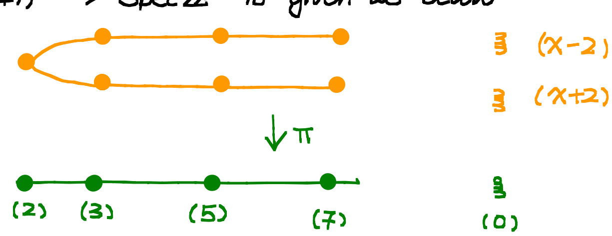
If I is not a prime $\Rightarrow \exists\, a, b \notin I$, $ab \in I$. $\Rightarrow V(I) = V(I+(a)) \cup V(I+(b))$
$= (V(I) \cap V(a)) \cup (V(I) \cap V(b))$ but $V(a) \cap V(I) \neq V(I)$, $V(b) \cap V(I) \neq V(I)$
since $a, b \notin I$. Hence $V(I)$ is not irreducible.

Conversely if I is prime. $V(I) = \overline{(I)}$ $\square$

Examples:

(1). $\operatorname{Spec}(\mathbb{Z}[x]/(x^2-4))$ revisited.

$\operatorname{Spec}(\mathbb{Z}[x]/(x^2-4)) \xrightarrow{\pi} \operatorname{Spec} \mathbb{Z}$ is given as below

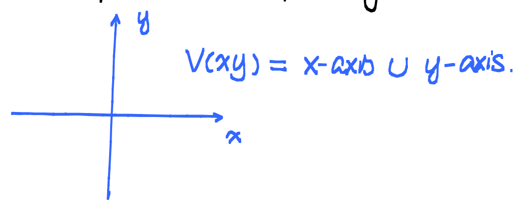


There are 2 irreducible components, corresponding to two minimal primes i.e. $\overline{\{(x-2)\}}$ and $\overline{\{(x+2)\}}$.

(2). k : a field. What are the irreducible components of $\mathrm{Spec}(k[x,y]/(xy))$?
(or what are the irreducible components of $V(xy)$?)

$\mathfrak{p} \supseteq (xy) \Rightarrow xy \in \mathfrak{p} \Rightarrow$ either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. $\Rightarrow \mathfrak{p} \supseteq (x)$ or $\mathfrak{p} \supseteq (y)$

Thus (x), (y) are the minimal primes, corresponding to the irreducible components



(3). As a generalization, what are the irreducible components of $\mathrm{Spec}\, R$, where $R = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/I$, and I is generated by the entries of the 2x2 matrix $X \cdot Y = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$?

To find the irreducible components, we shall use a property of R to be introduced later, that it is Jacobson. Then to find its irreducible components, it suffices to find the irreducible components of $\max\mathrm{Spec} R$.

Define a $GL(2,k) \times GL(2,k) \times GL(2,k)$ action on $X \cdot Y$ by $g_1 X g_2 g_2^{-1} Y g_3$. Then since $GL(2,k) \times GL(2,k) \times GL(2,k)$ is irreducible, so is its image in $\mathbb{A}^4 \times \mathbb{A}^4$. In particular, this action preserves $\{X, Y \mid XY = 0\}$. It suffices to classify the orbits of this set under the group action.

Now, by simple linear algebra, we know that the orbits of $GL(2,k) \times GL(2,k)$ on $M_2(k)$ by $g_1 X g_2$ are $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$. Now we discuss case by case:

a. $g_1 X g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $0 = g_1 X g_2 g_2^{-1} Y g_3 = (g_2^{-1} Y) g_3 \Rightarrow Y = 0$. This gives us the irreducible component $Y = 0$, or the ideal $(y_{11}, y_{12}, y_{21}, y_{22})$, since $\{X = g_1^{-1} g_2^{-1} \mid g_1, g_2 \in GL(2,k)\} \cong GL(2,k)$ is Zariski dense in $M_2(k)$.

b. $g_1 X g_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $0 = X$, Y arbitrary. Hence the ideal is $(x_{11}, x_{12}, x_{21}, x_{22})$.

c. $g_1 X g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In this case more care is needed. Let's call this component(s) Z, and we will show that Z is actually irreducible. To do this, it suffices to show that $Z \setminus \{0\}$ is irreducible, where 0 is the closed point $(\underline{x}, \underline{y})$. We will show that $Z \setminus \{0\}$ is covered by 2 open neighborhoods U, V s.t. $U, V, U \cap V$ are irreducible, the result then follows.

§2. Images and Ring Maps of Finite Presentation.

Some topology:

Def: (a). We say a topological space is *quasi-compact* iff every open covering has a finite refinement.

(b). A continuous map $f: X \to Y$ is called *quasi-compact* iff $f^{-1}(V)$ is quasi-compact for every open quasi-compact V in Y.

(c). We say that $Z \hookrightarrow X$ is *retro-compact* iff the map $Z \hookrightarrow X$ is quasi-compact.

Remark: Often a map $f: X \to Y$ is called proper iff $f^{-1}(K)$ is compact whenever K is compact. (In particular, in T_1 spaces, points are compact, thus fibers are compact if f is proper.) Note the difference in requiring K.

Def: $f: X \to Y$ map of topological spaces.

(a). f is closed iff the image of every closed set is closed.

(b). f is universally closed iff $\forall$ topological space Z, the map $Z \times X \to Z \times Y$ is closed

(c). f is quasi-proper iff $f^{-1}(K)$ is quasi-compact whenever $K \subseteq Y$ is quasi-compact.

(Fact: quasi-proper $\Leftrightarrow$ universally closed $\Leftrightarrow$ f is closed and $\forall y \in Y$, $f^{-1}(y)$ is quasi-compact).

Ex. (1). Prove or find reference for the fact

(2). Explain what this means for locally compact Hausdorff spaces

(3). What's the relation between (b) and the condition "$\forall X \to Y$, $Z \to Y$, the induced map $X \times_Y Z \to Z$ is closed" ?

Def. A subset $E \subseteq X$ is called constructible if it's a finite union of subsets of the form $U \cap V^c$ where $U, V \subseteq X$ are open and retrocompact.

Lemma: The collection of constructible subsets is closed under finite intersections, finite unions and taking complements.

Pf: Suppose E is constructible, $E = \bigcup_{i=1}^{n} U_i \cap V_i^c$

Then $E^c = \bigcap_{i=1}^{n} (U_i^c \cup V_i) = \bigcup_{k=0}^{n} U_{i_1}^c \cap \cdots \cap U_{i_k}^c \cap V_{j_{k+1}} \cap \cdots \cap V_{j_n}$ $(1 \le i_1 < i_2 < \cdots < i_k \le n,\ 1 \le j_{k+1} < \cdots < j_n \le n)$

$= \bigcup_{k=0}^{n} (U_{i_1} \cup \cdots \cup U_{i_k})^c \cap (V_{j_{k+1}} \cap \cdots \cap V_{j_n})$

$E_1 \cup E_2 = \bigcup_{i=1}^{n} U_i \cap V_i^c \cup \bigcup_{i=n+1}^{m} U_i \cap V_i^c = \bigcup_{i=1}^{m} U_i \cap V_i^c$

$E_1 \cap E_2 = (\bigcup_{i=1}^{n} U_i \cap V_i^c) \cap (\bigcup_{j=1}^{m} U_j \cap V_j^c) = \bigcup_{i,j} U_i \cap U_j \cap V_i^c \cap V_j^c$

Thus it suffices to show that finite union and intersection of retrocompact sets are still retrocompact. Take W quasi-compact open in X.

$(U \cap V) \cap W = U \cap (V \cap W)$ $\quad$ $V \cap W$ quasi-compact (q.c.) since V retro-compact (r.c.)

$= U \cap (q.c.) = (q.c.)$

$(U \cup V) \cap W = (U \cap W) \cup (V \cap W) = (q.c.) \cup (q.c.) = (q.c.)$ $\square$

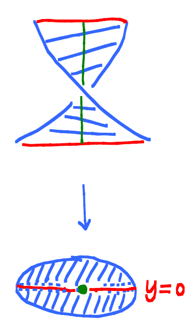
Example: (Back to commutative algebra)

$\mathbb{C}[x,y] \longrightarrow \mathbb{C}[u,y] \quad x \mapsto uy \quad y \mapsto y$

What's the image?

$\mathbb{C}^2 \longleftarrow \mathbb{C}^2$

$(uy, y) \longleftarrow\!\shortmid (u,y)$



blow-up in one coordinate chart: with one line deleted

•: (0,0)

$\Rightarrow$ image doesn't contain $y=0$ except $(0,0)$

$\Rightarrow$ image $= \underbrace{D(y)}_{\text{open (q.c.)}} \amalg \underbrace{\{\text{the maximal ideal } (x,y)\}}_{\text{closed} = \text{open}^c\text{, and in Noetherian spaces open sets are q.c.}}$

Lemma:

R: a ring, $U \subseteq \operatorname{Spec} R$ open. TFAE:

(a). U retro-compact

(b). U is quasi-compact

(c). U is a finite union of standard open sets.

Pf: (a) $\Rightarrow$ (b) : Spec R is q.c.

(b) $\Rightarrow$ (c) : Any open set is the union of standard open sets

(c) $\Rightarrow$ (a) : Enough to show that $D(f) \subseteq \operatorname{Spec} R$ is q.c.

Take W q.c. by (b) $\Rightarrow$ (c), $W = \bigcup_{j=1}^{m} D(g_j) \Rightarrow D(f) \cap W = \bigcup_{j=1}^{m} D(g_j f) = \bigcup_{j=1}^{m} \operatorname{Spec} R_{g_j f}$

Thus finite union of q.c. is still q.c. $\square$

Examples:

$R = \mathbb{C}[x_1, \cdots, x_n, \cdots]$, $U = \bigcup_{j=1}^{\infty} D(x_j)$ is not q.c.

$U = \operatorname{Spec} R \setminus \{(x_1, \cdots, x_n, \cdots)\}$

Take the open cover of $\{D(x_j)\}$ itself, then the point $(0, 0, \cdots, \overset{j\text{th}}{1}, 0, 0, \cdots)$ is only in U_j but not any other U_i $(i \neq j)$. Thus this cover has no finite subcover.

This is because, there is no finitely generated ideal whose radical is the maximal ideal $(x_1, \cdots, x_n, \cdots)$, which is the content of the next:

Lemma: $\operatorname{Spec} R \setminus V(I)$ is not quasi-compact $\Leftrightarrow$ there do not exist finitely many $f_1, \cdots, f_r \in R$ s.t. $\sqrt{(f_1, \cdots, f_r)} = \sqrt{I}$.

We can't change the condition in the lemma to "I being finitely generated" since we have the counter-example:

Example: (Spec$R$ Noetherian $\not\Rightarrow$ R Noetherian.)

$\mathbb{C}[x_1, \cdots, x_n, \cdots] / (x_1, x_2^2, x_3^3, \cdots)$ $I = (\bar{x}_1, \bar{x}_2, \cdots)$ $\sqrt{I} = \sqrt{0}$ and I is not finitely generated, yet the spectrum is a singleton thus q.c.

Pf of lemma:

$\operatorname{Spec} R \setminus V(I)$ not q.c. If there were such $f_1, \cdots f_r \Rightarrow V(f_1, \cdots, f_r) = V(I)$

$\Rightarrow \bigcup_{i=1}^{r} D(f_i) = \operatorname{Spec} R \setminus V(I)$ but the L.H.S. is q.c

Conversely if $\operatorname{Spec} R \setminus V(I)$ q.c. $\Rightarrow$ it's the union of finitely many $D(f_i)$ $i = 1, \cdots, r$

$\Rightarrow V(f_1, \cdots, f_r) = V(I) \Rightarrow \sqrt{(f_1, \cdots f_r)} = \sqrt{I}$. $\square$

Lemma:

$R \xrightarrow{\varphi} S$ a map of rings, then $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is a quasi-compact map of topological spaces.

Pf: This is because, the preimage of any $D(f)$, $f \in R$ is the standard open set $D(\varphi(f))$. $\square$

Lemma:

Let R be a ring and let $T \subseteq \operatorname{Spec} R$ be a constructible set, then there exists a finitely presented R-algebra S s.t. the image of $\operatorname{Spec} S \to \operatorname{Spec} R$ is T.

Pf: It suffices to write $U \cap V^c = (\bigcup_{j=1}^{m} D(f_j)) \cap V(g_1, \cdots, g_r)$ as the image of $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ for some finitely presented S.

Now, $U \cap V^c = \bigcup_{j=1}^{m} (D(f_j) \cap V(g_1, \cdots, g_m)) = \bigcup_{j=1}^{m} \operatorname{Spec}(R_{f_j} / (g_1, \cdots, g_m) R_{f_j})$

Thus we may take $S = \bigoplus_{j=1}^{m} (R/(g_1, \cdots, g_m))_{f_j} = \bigoplus_{j=1}^{m} R[x]/(xf_j - 1, g_1, \cdots, g_m)$,

which is a finitely presented R algebra. $\square$

Lemma:

R a ring, $f \in R$. Then the image in $\operatorname{Spec} R$ of a constructible set in $\operatorname{Spec} R_f$ is constructible.

Pf: Let U, V be quasi-compact open and $T = U \cap V^c$, we have to show that the image of T in $\operatorname{Spec} R_f$ is constructible.

Now
$$\begin{tikzcd} \operatorname{Spec} R_f \arrow[r, hook] \arrow[dr, phantom, "\cong"] & \operatorname{Spec} R \\ U, V \arrow[u, phantom, sloped, "\subseteq"] & D(f) \arrow[u, hook] \end{tikzcd}$$
, $U, V \longrightarrow U', V'$ (homeomorphism)

$\Rightarrow \operatorname{im} T = U' \cap V'^c$

It suffices to show that U', V' are retrocompact in $\operatorname{Spec} R$ or equivalently U', V' are a finite union of standard open sets. But this is clear since standard open sets in $\operatorname{Spec} R_f$ is a subclass of standard open sets of $\operatorname{Spec} R$. $\square$

Lemma:

R a ring, $I \subseteq R$ is a finitely generated ideal. Then the image of any constructible set of $\operatorname{Spec} R/I$ is constructible in $\operatorname{Spec} R$.

Pf: Let $I=(f_1,\cdots,f_m)$, then $V(I)=\bigcap_{j=1}^{m} V(f_j) \cong \operatorname{Spec} R/I$.
Any constructible set T in $\operatorname{Spec} R/I$ is of the form $U\cap V^c$, U, V quasi-compact open. It suffices to show that $U=U'\cap V(I)$, $V=V'\cap V(I)$, where U', V' are quasi compact open subsets of $\operatorname{Spec} R$, then

$$\operatorname{im} T = U'\cap V(I)\cap (V'\cap V(I))^c = U'\cap V(I)\cap (V'^c\cup V(I)^c)$$
$$= U'\cap \Big(V'\cup \bigcup_{j=1}^{m} D(f_j)\Big)^c \quad \text{and} \quad V'\cup \bigcup_{j=1}^{m} D(f_j) \text{ is quasi-compact.}$$

Indeed, if $U=\bigcup_{k=1}^{m} D(\bar g_k)$ where $\bar g_i\in R/I$ $\Rightarrow$ we may take $U'=\bigcup_{k=1}^{m} D(g_k)$ where g_k is a lifting of $\bar g_k$ in R. □

Lemma:
R a ring, then the map $\operatorname{Spec} R[x]\longrightarrow \operatorname{Spec} R$ is open, and the image of any standard open is quasi-compact open.
Pf: The second statement implies the first.
Take $f\in R[x]$, $f=a_n x^n+\cdots+a_0$, then we have the diagram:

$$\begin{tikzcd}
R[x]_f \arrow[r] & \kappa(\mathfrak p)\otimes_R R[x]_f = \kappa(\mathfrak p)[x]_{\bar f} \\
R \arrow[u, "\varphi"] \arrow[r] & \kappa(\mathfrak p) \arrow[u]
\end{tikzcd}$$

Thus $\mathfrak p$ is in the image of $\operatorname{Spec}\varphi \Leftrightarrow \kappa(\mathfrak p)[x]_{\bar f}$ is non-zero.
$\Leftrightarrow \bar f$ is not 0 in $\kappa(\mathfrak p)[x]$ since $\kappa(\mathfrak p)[x]$ is a domain
$\Leftrightarrow$ The coefficients of f mod $\mathfrak p$ are not all 0
$\Leftrightarrow \mathfrak p \not\supseteq (a_n,\cdots,a_0)$
$\Leftrightarrow \mathfrak p\in \bigcup_{i=1}^{n} D(a_i)$
Thus $\operatorname{im} D(f)=\bigcup_{i=1}^{n} D(a_i)$, which is quasicompact open in $\operatorname{Spec} R$. □

Lemma:
Let R be a ring, $f, g\in R[x]$. Assume the leading coefficient of g is a unit, then there exist $r_i\in R$, $i=0,\cdots,d-1$, s.t. $\operatorname{Im}(D(f)\cap V(g))=\bigcup_{i=0}^{d-1} D(r_i)$ under the map $\operatorname{Spec} R[x]\longrightarrow \operatorname{Spec} R$.
Pf: Write $g(x)=ux^d+a_{d-1}x^{d-1}+\cdots+a_0$, and consider $A=R[x]/(g)$, since u is a unit, $A\cong R^{\oplus d}$ as R-modules.

$D(f) \cap V(g) = \operatorname{Spec}(A_{\bar{f}})$ where $\bar{f} = f \mod (g)$.

By Euclidean algorithm, we may assume $\deg f < \deg(g)$, and consider the R linear map of A: $A \longrightarrow A$, $\mathfrak{z} \mapsto f \cdot \mathfrak{z}$. Then this map has a characteristic polynomial $P(T)$ and $P(f) = f^d + r_{d-1} f^{d-1} + \cdots + r_0 = 0$. Consider the diagram:

$$\begin{tikzcd} A_{\bar{f}} \arrow[r] & \kappa(\mathfrak{p}) \otimes_R A_{\bar{f}} = \left(\frac{\kappa(\mathfrak{p})[x]}{(g)}\right)_{\bar{f}} \\ R \arrow[u] \arrow[r] & \kappa(\mathfrak{p}) \arrow[u] \end{tikzcd}$$

Now, $\mathfrak{p} \in \operatorname{Im}(D(f) \cap V(g)) \Leftrightarrow \left(\frac{\kappa(\mathfrak{p})[x]}{(g)}\right)_{\bar{f}} \neq 0$

$\Leftrightarrow$ $\bar{f}$ is not nilpotent in $\frac{\kappa(\mathfrak{p})[x]}{(g)}$

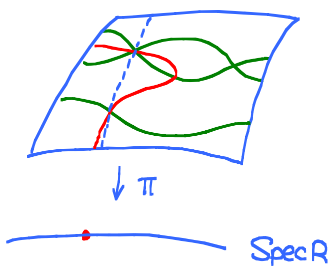
$\Leftrightarrow$ The characteristic polynomial of $(u \mapsto \bar{f} \cdot u)$ is not 0 in $\kappa(\mathfrak{p})[x]/(g) \cong \kappa(\mathfrak{p})^{\oplus d}$

$\Leftrightarrow$ $P(T) \neq T^d \mod \mathfrak{p}$

$\Leftrightarrow$ $\mathfrak{p} \not\supseteq (r_0, \cdots, r_{d-1})$

$\Leftrightarrow$ $\mathfrak{p} \in \bigcup_{i=0}^{d-1} D(r_i)$ $\square$

Geometrically:



: $g(x) = 0$

: $f(x) = 0$

If f passes through all points of $F \cap V(g)$ where F is a fiber, then $\pi(F) \notin \operatorname{im}$.

Thm: (Chevalley)

Assume $R \longrightarrow S$ is of finite presentation, then $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ preserves constructible sets.

Pf: We do reduction of the ring map. $S = R[x_1, \cdots, x_n] / (f_1, \cdots, f_m)$

We have shown that $\operatorname{Spec}(\varphi)$ preserves constructible sets, it thus suffices to show that $\operatorname{Spec}(\psi)$ preserves constructible sets.

$$\begin{tikzcd} S \\ R[x_1, \cdots, x_n] \arrow[u, "\varphi"] \\ R[x_1, \cdots, x_{n-1}] \arrow[u] \\ \vdots \arrow[u] \\ R[x_1] \arrow[u] \\ R \arrow[u, "\psi"] \end{tikzcd}$$

Furthermore, we are reduced to prove that the image of $U \cap V^c$ is constructible under $\operatorname{Spec}(\psi): \operatorname{Spec} R[x] \longrightarrow \operatorname{Spec} R$ where $U = \bigcup_{i=1}^{n} D(f_i)$ $V = \bigcup_{j=1}^{m} D(g_j)$. $f_i, g_j \in R[x]$

$U \cap V^c = (\bigcup_{i=1}^{n} D(f_i)) \cap V(\Sigma(g_i)) = \bigcup_{i=1}^{n} (D(f_i) \cap V(g_1, \cdots, g_m))$. Hence it suffices to prove for $D(f) \cap V(g_1, \cdots, g_m) = E$

Now suppose $c \in R$. Then we have:

$$\begin{array}{ccccc} E \subseteq \operatorname{Spec} R[x] & = & \operatorname{Spec} R_c[x] & \coprod & \operatorname{Spec} \frac{R[x]}{cR[x]} \\ \downarrow \pi & & \downarrow \pi_1 & & \downarrow \pi_2 \\ \operatorname{Spec} R & = & \operatorname{Spec} R_c & \coprod & \operatorname{Spec} R/cR \end{array}$$

If E is constructible, let $E_1 = E \cap \operatorname{Spec} R_c[x]$, $E_2 = E \cap \operatorname{Spec} \frac{R}{(c)}[x]$. It suffices to show that $\pi_1(E_1)$ and $\pi_2(E_2)$ are constructible, since by previous lemmas, closed and open immersions preserve constructible sets. Note that:

$E_1 = D(f') \cap V(g_1', \cdots, g_m')$, where f', g_i' are the images of f, g_i in $R_c[x]$;

$E_2 = D(\bar{f}) \cap V(\bar{g}_1, \cdots, \bar{g}_m)$, where $\bar{f}$, $\bar{g}_i$ are the images of f, g_i in $\frac{R}{(c)}[x]$.

Number g_i's so that $\deg g_i \leq \deg g_{i+1}$. We prove by induction on $m + \Sigma \deg g_i = N$

$N=1$: $m=1$, $\deg g_1 = 0$. Take $R \ni c = g_1$. $E_1 = \Phi$. $E_2 = D(\bar{f})$, the result follows from lemma for closed immersions.

Suppose the result is true for $N<k$

$N=k$: let c be the leading coefficient of g_1. Then $\pi_2(E_2)$ is constructible since now $\deg \bar{g}_1 < \deg g_1 \Rightarrow \Sigma \deg \bar{g}_i + m < N$.

For $\pi_1(E_1)$, now g_1 is a polynomial with leading coefficient a unit. Thus we may either write $g_m = h \cdot g_1 + r$ with $\deg r < \deg g_1$ or $g_m = h g_1$ so $E_2 = D(\bar{f}) \cap V(\bar{g}_1, \cdots, \bar{g}_{m-1})$

In any case N decreases and by induction hypothesis we are done. $\square$

Example: (What this means in number theory?)

$0 \longrightarrow \mathbb{Z} \longrightarrow R$: finite type.

Chevalley's thm $\Rightarrow$ $\operatorname{Im}(\operatorname{Spec} R)$ in $\operatorname{Spec} \mathbb{Z}$ is constructible. But the sets of the form $U \cap V^c$ in $\operatorname{Spec} \mathbb{Z}$ are either (i) finite set of closed points or (ii) non empty open subsets of $\operatorname{Spec} \mathbb{Z}$.

Claim: $\operatorname{Im}(\operatorname{Spec} R)$ is not a finite closed set.

Indeed, otherwise $V(n \cdot 1_R) = \operatorname{Spec} R$, i.e. n is nilpotent in R, but this is impossible since we assumed that $\mathbb{Z}$ injects into R.

Conclusion: $\forall$ all but finitely many primes $\mathfrak{p}$ in $\mathbb{Z}$, $\exists\, \mathfrak{q} \in \operatorname{Spec} R$ s.t. $\mathfrak{q} \cap \mathbb{Z} = \mathfrak{p}$.

§3. Noetherian Rings

Basic fact: R Noetherian

$\Leftrightarrow$ Every ideal I of R is finitely generated.

$\Leftrightarrow$ Ascending chain condition (a.c.c.) for ideals of R.

Lemma:

Any finitely generated ring over a Noetherian ring is Noetherian.

Pf: Let R be Noetherian. The following two facts follow from the relation between the set of ideals in R and the set of ideals in R/I and $S^{-1}R$.

(1). If $I \subseteq R$ is an ideal, then R/I is Noetherian

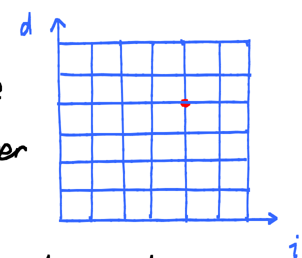
(2). If S is a multiplicative set, then $S^{-1}R$ is Noetherian.

Along with the next fact, these facts together prove the lemma.

(3) $R[x]$ is Noetherian.

Suppose $J_1 \subseteq J_2 \subseteq \cdots$ is a chain of ideals in $R[x]$. Set $I_{i,d} = \{0\} \cup \{$leading coefficients of polynomials of deg d in $J_i\}$, which is obviously an ideal. Moreover we have $i \le i'$, $d \le d' \Rightarrow I_{i,d} \subseteq I_{i',d'}$.

Claim: there are at most finitely many distinct ideals $I_{i,d}$ among the collection of ideals $\{I_{i,d}\}_{(i,d)\in\mathbb{N}\times\mathbb{N}}$.



In fact, for each fixed i, $\{I_{i,1} \subseteq I_{i,2} \subseteq \cdots\}$ so by the Noetherian hypothesis, $\exists$ maximal element I_{i,d_i} s.t. whenever $d' \ge d_i \Rightarrow I_{i,d'} = I_{i,d_i}$. Moreover, we have $I_{i,d_i} \subseteq I_{i+1,d_{i+1}}$, since $I_{i+1,d_{i+1}} = I_{i+1,d_{i+1}+k}$, $\forall k \ge 0$ and we may choose such k that $d_{i+1}+k \ge d_i$.

Again by the Noetherian hypothesis, $\exists$ maximal element in the chain $\{I_{1,d_1} \subseteq I_{2,d_2} \subseteq \cdots\}$, say $I_{N_1,d_{N_1}}$, and then we conclude that there are only finitely many distinct ones for $d \ge d_{N_1}$. Furthermore, below $d = d_{N_1}$, each horizontal chain is bounded eventually and thus with finitely many distinct ones. Now take I_{N,d_N} so that all distinct types of ideals lie on the lower left hand side of it.

Claim: $J_N = J_{N+1} = \cdots\cdots$

If $f \in J_M$, $M \ge N$, we argue by induction on $\deg f$ that $f \in J_N$.

$\deg f = 0 \Rightarrow f = r \in I_{M,0} = I_{N,0} \Rightarrow r \in J_N$.

$\deg f > 0$. Now suppose we have dealt with all cases where $\deg f < k$.
Now $\deg f = k$. Since N is so choosen we know that $I_{M,k} = I_{N,k}$, and thus $\exists g \in J_N$ $\deg g = k$ and g has the same leading coefficient as f. $\Rightarrow f-g \in J_M$ has degree $< k$. By induction hypothesis, $f-g \in J_N \Rightarrow f \in J_N$. $\square$

Since any field or principle ideal domain is Noetherian, we have:

Cor.

Any finite type algebra over a field k or $\mathbb{Z}$ is Noetherian. $\square$

Cor.

Any finite type algebra over a Noetherian ring R is of finite presentation over R.

Pf: Since $R[x_1, \cdots, x_n]$ is also Noetherian, any ideal of it is finitely generated. $\square$

Def: A topological space X is Noetherian if the descending chain condition (d.c.c.) holds for all closed subsets of X.

Lemma:

If R is a Noetherian ring, then $\operatorname{Spec} R$ is a Noetherian topological space. $\square$

The converse is not true:

Example: A Noetherian topological space whose ring is not Noetherian.

Take $R = k[x_1, x_2, \cdots]/(x_1, x_2^2, x_3^3, \cdots)$. Then $\operatorname{Spec} R = \{pt\}$ and thus Noetherian. But R is not since $\mathfrak{m} = (\bar{x}_1, \bar{x}_2, \cdots)$ is not finitely generated.

Lemma:

Let X be a noetherian topological space.

(1). Any $T \subseteq X$ is Noetherian.

(2). X has finitely many irreducible components.

(3). Any irreducible component of X contains a non-empty open subset of X.

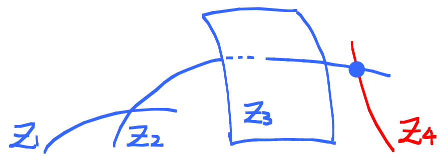
Pf: (1). Any descending chain of closed subsets is of the form $Z_1 \cap T \supseteq Z_2 \cap T \supseteq \cdots$ where Z_i's are closed subsets of X. Then we can form a descending chain of closed subsets of X with the same intersection as $Z_i \cap T$, namely $Z_1 \supseteq Z_1 \cap Z_2 \supseteq Z_1 \cap Z_2 \cap Z_3 \supseteq \cdots$

and it eventually stablizes by Noetherian hypothesis. $\Rightarrow \{Z_i \cap T\}$ eventually stablizes.

(2). Let $\mathcal{S} = \{$closed subsets of X which do NOT have finitely many irreducible components$\}$

If $\mathcal{S}$ were not empty, by Noetherian hypothesis, it would contain a minimal element Z. Z couldn't be irreducible since $Z \in \mathcal{S} \Rightarrow Z = Z' \cup Z''$, Z', Z'' closed and not equal to $Z \Rightarrow Z', Z'' \notin \mathcal{S}$ by minimality of $Z \Rightarrow Z' = \bigcup_{i=1}^{n} Z_i$, $Z'' = \bigcup_{j=1}^{m} Z_j$ with Z_i, Z_j irreducible $\Rightarrow Z = \bigcup_{i=1}^{n} Z_i \cup \bigcup_{j=1}^{m} Z_j$, contradiction.

(3). If $X = Z_1 \cup \cdots \cup Z_m$, then for each Z_i, take $Z_i \setminus (\bigcup_{j \neq i} Z_j)$



$Z_4 \setminus (Z_1 \cup Z_2 \cup Z_3)$ is open. □

Rmk: The argument in (2) is called "Noetherian induction"

Since we know that there is a 1-1 correspondence between minimal primes of a ring R and the irreducible components of $\operatorname{Spec} R$

Cor. If R is a Noetherian, then R has only finitely many minimal primes. Thus $\sqrt{(0)} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, intersection over these minimal primes. □

- Artin-Rees Lemma.

Lemma: R: Noetherian

(a). Any finite R-module is of finite presentation

(b). Any submodule of a finite R-module is finite.

Pf: (b) $\Rightarrow$ (a): If M is finite $\Rightarrow M$ is finitely presented: $M = R^{\oplus n}/N$ for some n and $N \subseteq R^{\oplus n}$. Now by (b), N is finite, i.e. $N = R^{\oplus m}/K$ for some m and $K \subseteq R^{\oplus m}$ $\Rightarrow R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$ is a finite presentation of M as an R-module.

Thus it suffices to prove (b).

Let $N \subseteq M$ be a submodule. We will prove that N is finitely generated by induction on the number of generators of M.

M is generated by 1 element: $M \cong R/I \supseteq N \Rightarrow N \cong J/I$ for some $I \subseteq J \subseteq R$. Since R is Noetherian, J is finitely generated, then so is $J/I \cong N$.

Induction step: Suppose M is generated by $x_1, \cdots, x_n$. Let M' be the submodule generated by $x_1, \cdots, x_{n-1}$ $\Rightarrow$ M/M' is generated by 1 element, namely x_n.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N\cap M' & \longrightarrow & N & \longrightarrow & N/N\cap M' & \longrightarrow & 0 \end{array}$$

By induction hypothesis, $N\cap M'$ and $N/N\cap M'$ are finitely generated. Chose a set of generators $\{x_1, \cdots, x_k\}$ for $N\cap M'$ and $\{\bar{y}_1, \cdots, \bar{y}_\ell\}$ for $N/N\cap M'$, where y_j's are a lift of $\bar{y}_j$ in N. Then N is generated by $\{x_1, \cdots, x_k, y_1, \cdots, y_\ell\}$

Indeed, take any $z \in N$, $\bar{z}$ in $N/N\cap M'$ can be written as $\bar{z} = \Sigma r_j \bar{y}_j$. Thus $z - \Sigma r_j y_j \mapsto 0$ in $N/N\cap M'$, and must be in $N\cap M'$. Therefore $z - \Sigma r_j y_j = \Sigma s_i x_i$ $\Rightarrow z = \Sigma s_i x_i + \Sigma r_j y_j$. $\square$

Lemma (Artin-Rees Lemma)

R: Noetherian, $I \subseteq R$ is an ideal and $N \subseteq M$, finitely R-modules. Then: $\exists\, c \in \mathbb{N}$, s.t. $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n \geq c$.

Pf: Consider the ring $S = R \oplus I \oplus I^2 \oplus \cdots = \oplus_{n\geq 0} I^n$, where the multiplication is given by $I^n \times I^m \longrightarrow I^{n+m}$, $(a, b) \mapsto ab$.

Suppose $I = (f_1, \cdots, f_n)$, then define $\psi: R[x_1, \cdots, x_n] \longrightarrow S$ by $\psi(x_i) = f_i \in I$. Since

I^m = ideal generated by products of the form $a_1 \cdots a_n$, $a_i \in I$

= ideal generated by products of the form $f_{i_1} \cdots f_{i_n}$, $1 \leq i_j \leq n$

Therefore, deg m part of $R[x_1, \cdots, x_n]$ surjects onto I^m via ψ, and thus S is Noetherian. Similarly, consider the module $\tilde{M} = M \oplus IM \oplus I^2M \oplus \cdots = \oplus_{n\geq 0} I^n M$ as an S-module $S \times \tilde{M} \longrightarrow \tilde{M}$ ($I^n \times I^m M \longrightarrow I^{n+m}M$, multiplication given in the obvious fashion.

Claim: $\tilde{M}$ is a finite S-module. Namely, if $x_1, \cdots, x_n \in M$ generate M as an R-module, then they also generate $\tilde{M}$ as an S-module.

Consider the S-submodule $\tilde{M}'$ of $\tilde{M}$ given by $\tilde{M}' = \oplus_{n\geq 0} I^n M \cap N \subseteq \tilde{M}$.

By the previous lemma, $\tilde{M}'$ is also finitely generated as an S-module, say by $\xi_j \in \tilde{M}$. By taking their homogeneous components, we may assume that each ξ_j is homogeneous in $I^{n_j} M \cap N$ for some n_j, $j = 1, \cdots, N$.

Let $c \triangleq \max\{n_j \mid j=1,\cdots,n\}$. $\forall\, n \geqslant c$, and $\xi \in I^n M \cap N$

We may write $\xi = \sum s_j \xi_j$. Since the homogeneous components of s_j other than those in $I^{n-n_j}M \cap N$ will be canceled anyway, we may assume that $s_j \in I^{n-n_j}M \cap N$. Moreover, since $c \geqslant n_j$, $\forall j$. $I^{n-n_j} = I^{n-c} I^{c-n_j}$,

$\Rightarrow$ $\xi \in \sum I^{n-n_j}(I^{n_j}M \cap N) \subseteq I^{n-c}(I^c M \cap N)$.

The other inclusion $I^{n-c}(I^c M \cap N) \subseteq I^n M \cap N$ is obvious. $\square$

§4. Jacobson Rings

In many cases rings will be Jacobson rings and the spectra will be Jacobson spaces i.e. if $X_0 = \{\text{closed points of } X\}$, X_0 is "very dense" in X i.e. Z closed in $X \Rightarrow Z = \overline{Z \cap X_0}$.

- Thm (Hilbert Nullstellensatz)

Let k be a field

(1). Any maximal ideal $m \subseteq k[x_1, \cdots, x_n]$ has its residue field $\kappa(m)$ a finite field extension of k

(2). Any radical ideal $I \subseteq k[x_1, \cdots, x_n]$ is the intersection of the maximal ideals containing I.

Furthermore, the same holds in any finite type k-algebra in place of $k[x_1, \cdots, x_n]$.

Pf: We will prove (1) by induction on n.

$n=0$ is clear

Suppose $m \subseteq k[x_1, \cdots, x_n]$ is maximal. Let $\mathfrak{p}$ be the contraction of m in $k[x_n]$.

Case I: $\mathfrak{p} \neq 0 \Rightarrow \mathfrak{p}$ is maximal, generated by a monic irreducible $p \in k[x_n]$ since $k[x_n]$ is a P.I.D. Then $k' \triangleq k[x_n]/\mathfrak{p}$ is a finite field extension of k. And we have a surjection $k'[x_1, \cdots, x_{n-1}] \xrightarrow{\psi} \kappa(m)$ given by $\psi(x_i) = \overline{x}_i$ in $\kappa(m)$ and $\psi([x_n]) = \overline{x}_n$ ($[x_n]$ standing for the class of x_n in k').

By induction hypothesis applied to $k'[x_1, \cdots, x_{n-1}]$, $\ker\psi$ is a maximal ideal and $\kappa(m)$ is a finite field extension of k'. Since k' is also finite over k, we deduce that $\kappa(m)$ is finite over k.

Case II. $\mathfrak{p} = 0$. Consider the ring extension $k[x_n] = k[x_n]/(0) \longrightarrow k[x_1, \cdots, x_n]/m = \kappa(m)$

$\Rightarrow \operatorname{Spec}\kappa(m) \xrightarrow{\varphi} \operatorname{Spec} k[x_n]$. Since the ring map is finitely presented (m being finitely generated since $k[x_1, \cdots, x_n]$ is Noetherian), by Chevalley's thm, the image of φ, i.e. the generic point of $\operatorname{Spec} k[x_n]$, is constructible, hence is non-empty and of the form $\bigcup_{i=1}^{n} U_i \cap V_i^c$, U_i, V_i open and retrocompact (q.c.). If $(0) \in V_i^c \Rightarrow V_i^c \supseteq \overline{(0)} = \operatorname{Spec} R$

$\Rightarrow \{\text{generic point}\}$ is non empty, open, thus would contain all but finitely many closed points, which is absurd. Thus $\mathfrak{p}$ can't be 0.

The last claim of the thm. is true since maximal ideals of $k[x_1,\cdots,x_n]/J$ are in 1–1 correspondence with maximal ideals containing J.

For (2), let $I \subsetneq k[x_1,\cdots,x_n]$ be a radical ideal, we have to show that $I = \bigcap_{m \supseteq I} m$ (= intersection of maximal ideals containing I).

Pick $f \notin I$, and we need to show that, $\exists$ a maximal ideal $m \supseteq I$, $f \notin m$. For this purpose, consider $D(f) \cap V(I) = \mathrm{Spec}((R/I)_f)$. Since I is radical, f is not nilpotent mod I $\Rightarrow D(f) \cap V(I) \neq \phi$. Pick a closed point $\bar{m}$ in it, which is a maximal ideal of $(R/I)_f$, and consider its preimage in R, denoted by m. Then we have $k \hookrightarrow R/m \hookrightarrow \kappa(\bar{m})$. By part (1), $\kappa(\bar{m})$ is finite over k. This implies that R/m is a field and m is a maximal ideal not containg f.

Similar as above, the case for any finite type k-algebra is true since any radical ideal of $k[x_1,\cdots,x_n]/J$ is a radical ideal of $k[x_1,\cdots,x_n]$ containing J. $\square$

Some Topology

Def: Let X be a topological space and $X_0 \subseteq X$ be the set of closed points. We say that X is Jacobson if every closed subset Z of X is the closure of $Z \cap X_0$.

A Counter Example:

$\mathrm{Spec}\,\mathbb{Z}_{(2)}$ (or any DVR) $= \{\eta, s\}$ where $\eta = (0)$ is the generic point and $s = (2)\mathbb{Z}_{(2)}$ is the only closed point. The topology on $\mathrm{Spec}\,\mathbb{Z}_{(2)}$ is given by $\{\phi, \{s\}, \{\eta, s\}\}$ (as closed sets). Thus $X_0 = \{s\}$ and $\overline{X \cap X_0} = \overline{\{s\}} = \{s\} \neq X$.

Remark: X Jacobson $\Rightarrow X_0 \to X$ induces a bijection of closed subsets of X and closed subsets of X_0 (with the induced topology). The same thing holds for open subsets (just take the complement). Thus it's equally well to do sheaf theory on X or X_0, and the Krull dimensions of X and X_0 are the same.

Lemma:

The irreducible closed subsets of X satisfy $Z = \overline{Z \cap X_0} \Rightarrow X$ Jacobson.

Pf: Take Z a closed subset of X, then $Z = \bigcup Z_i$, union of irreducible components.
$\Rightarrow Z_i = \overline{Z_i \cap X_0} \Rightarrow Z \supseteq \overline{Z \cap X_0} = \overline{\bigcup (Z \cap X_0)} \supseteq \bigcup \overline{Z \cap X_0} = \bigcup Z_i = Z$. $\square$

Lemma:
The property of X being Jacobson is local, i.e. $X = \bigcup_{i \in I} U_i$, U_i open, then X being Jacobson $\Leftrightarrow$ each U_i is Jacobson.
Pf: "$\Rightarrow$" $\forall$ Z closed subset of U, $Z = \overline{Z} \cap U$. X being Jacobson $\Rightarrow \overline{Z} = \overline{\overline{Z} \cap X_0}$
Moreover $\overline{Z} = (\overline{Z} \cap U^c) \cup \overline{Z} \cap U = (\overline{Z} \cap U^c) \cup Z$. Thus $\overline{\overline{Z}} = \overline{Z}$

Lemma:
X Jacobson. The following types of subsets T are Jacobson and $T_0 = T \cap X_0$
(1). open subsets
(2). closed subsets
(3). locally closed subsets
(4). finite union of locally closed subsets. ($\supseteq$ constructible subsets)

Back to algebra:

Def: R a ring. We say R is Jacobson if every radical ideal I is the intersection of maximal ideals containing it.

Cor. Any finite type algebra over a field is Jacobson. (by Nullstellensatz). $\square$

Lemma:

A ring R is Jacobson $\Leftrightarrow$ $\operatorname{Spec} R$ is a Jacobson space.

Pf: "$\Leftarrow$" Take a radical ideal $I \subseteq R$. Let $J = \bigcap_{m \supseteq I} m$. Then $J \supseteq I \Rightarrow V(J) \subseteq V(I)$ and $V(J)$ is the closure of $V(I) \cap (\operatorname{Spec} R)_0$ in $\operatorname{Spec} R$, but this is equal to $V(I)$. Moreover, J is a radical ideal since it's the intersection of radical ideals. $\Rightarrow V(J) = V(I)$ and $I = \sqrt{I} = \sqrt{J} = J$.

"$\Rightarrow$" If $Z = V(I) \subseteq \operatorname{Spec} R$ is a closed subset, we shall show that the set of closed is dense in Z, or in other words, every $D(f) \cap V(I)$ contains a closed point of Z, if $D(f) \cap V(I) \neq \phi$. Here we may assume I is radical and $f \notin I$. Since R is Jacobson, $I = \bigcap_{m \supseteq I} m \Rightarrow \exists m \supseteq I$ and $f \notin m \Rightarrow m \in D(f) \cap V(I)$. $\square$

Lemma:

$\mathbb{Z}$ is Jacobson.

Pf: Every non-0 radical ideal is of the form (n) where $m^2 \nmid n$, $\forall m, m^2 \neq 1$. Thus $n = \prod p_i$, p_i distinct $\Rightarrow (n) = \bigcap (p_i)$

Moreover, $0 = \bigcap (p_i)$ over all prime ideals. $\square$

Lemma:

Let $R \to S$ be a ring map, m a maximal ideal in R, and let q be an ideal lying over m. If $\kappa(m) \subset \kappa(q)$ is algebraic, then q is maximal.

Pf: Consider the diagram on the right:

$$\begin{tikzcd} S \arrow[r] & S/q \arrow[r, hook] & \kappa(q) \\ R \arrow[u] \arrow[r] & R/m = \kappa(m) \arrow[u] \arrow[ur, phantom, ""] & \end{tikzcd}$$

$\kappa(m) \subset \kappa(q)$ is algebraic, and $\kappa(m) \hookrightarrow S/q \hookrightarrow \kappa(q)$

$\Rightarrow S/q$ is a field.

$\Rightarrow q$ is maximal. $\square$

Prop: Let R be a Jacobson ring and let $R \longrightarrow S$ be of finite type, then:

(1). S is Jacobson.

(2). $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ maps closed points to closed points.

(3). If $m' \subseteq S$ is maximal lying over $m \subseteq R$, then the residue extension $k(m) \subseteq k(m')$ is finite.

Before giving the proof, we see that:

- Any finite type $\mathbb{Z}$-algebra is Jacobson
- If R is of finite type over $\mathbb{Z}$ and $m \subseteq R$ is a maximal ideal, then $\mathbb{Z} \cap m = (p)$ with p prime and $\mathbb{F}_p = \mathbb{Z}/p \longrightarrow k(m)$ is a finite field extension.

Pf of prop.

Write $S = R[x_1, \cdots, x_n]/I$, then we see that it suffices as before to prove the prop. for 2 cases: (i) $S = R/I$; (ii): $S = R[x]$

Case (i) is easy. since $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$ is a closed immersion thus $V(I)$ is Jacobson as a topological space, thus R/I is Jacobson. (2) and (3) follow.

Case (ii): By a top lemma. to show that $\operatorname{Spec} R[x]$ is Jacobson, it suffices to show for irreducible subsets of $\operatorname{Spec} R[x]$. Thus take q prime in $R[x]$, $\mathfrak{p} \triangleq q \cap R$. Take $f \in R[x]$, $f \notin q$, we need to show that $D(f) \cap V(q)$ contains a closed point.

Consider the diagram on the right. We are reduced to consider the case where $\mathfrak{p} = 0$ and R is a domain. By case (i) R is also Jacobson.

$$\begin{array}{ccc} R[x] & \longrightarrow & R/\mathfrak{p}[x] \supseteq \bar{q} \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/\mathfrak{p} \end{array}$$

(I). If $q = (0)$. Write $f = a_d x^d + \cdots + a_0$, since $f \notin q$, not all a_i are zero. Since R is Jacobson, we can find a maximal ideal m of R such that $a_i \notin m$ for some i. Thus $f \neq 0$ in $k(m) \otimes R[x] = k(m)[x]$, and we can choose a maximal ideal not containing the image of f. since $k(m)[x]$ is Jacobson by Hilbert Nullstellensatz, and thus the inverse image $m' \subseteq R[x]$ defines a closed point of $D(f) \cap V(q)$ and $k(m')$ is finite over $k(m)$, by Nullstellensatz again.

(II). If $q \neq (0)$. Then $0 \longrightarrow R \longrightarrow \frac{R[x]}{q} \rightsquigarrow 0 \longrightarrow K \longrightarrow K[x]/qK[x]$ where K is the fraction field of R. Since $K[x]$ is a P.I.D. $qK[x] \cong (g)$ for an irreducible polynomial

$g \in K[x]$. We may further assume $g = b_e x^e + \cdots + b_0$, $b_i \in R$, $b_e \neq 0$. Since R_{b_e} is also Jacobson, and $R_{b_e} \to R$, $R[x] \to R_{b_e}[x]$ induces isomorphisms on residue fields, we are reduced to the case where g is monic. Then $R[x]/\mathfrak{q} \cong R^{\oplus e} \subseteq K[x]/\mathfrak{q} \cong \oplus K^{\oplus e}$ $\Rightarrow$ $\mathfrak{q} = (g)$. By lemma ???, the image of $D(f) \cap V(g)$ in $\operatorname{Spec} R$ is $D(r_1) \cup \cdots \cup D(r_d)$ for some $r_i \in R$. Take any maximal ideal in this image, say $\mathfrak{m}$, and consider any prime $\mathfrak{m}'$ lying over $\mathfrak{m}$. Since $\mathfrak{m}' \supseteq g$, the residue field of $\mathfrak{m}'$ is finite over $\kappa(\mathfrak{m})$ thus $\kappa(\mathfrak{m}) \subseteq \frac{R[x]}{\mathfrak{m}'} \subseteq \kappa(\mathfrak{m}') \Rightarrow \mathfrak{m}'$ is maximal and $\kappa(\mathfrak{m}')$ is finite over $\kappa(\mathfrak{m})$

Thus we have shown that $\operatorname{Spec} R[x]$ is Jacobson, and it follows for $R[x]$. Moreover, take $\mathfrak{q}$ to be any maximal ideal of $R[x]$. The above argument shows that $\mathfrak{m}'$ is mapped to $\mathfrak{m}$ and $\kappa(\mathfrak{m}')$ is finite over $\kappa(\mathfrak{m})$. $\square$

Remarks:

In case R is Noetherian and Jacobson. (e.g. of finite type over k or $\mathbb{Z}$)

$R \to S$ finite type $\rightsquigarrow$ $\operatorname{Spec} S \xrightarrow{f} \operatorname{Spec} R$ the induced map on the spectrum.

Consider the diagram:

$$\begin{array}{ccc}
\{\text{Constructable subsets of } \operatorname{Spec} R\} & \longrightarrow & \{\text{Constructable subsets of } (\operatorname{Spec} R)_0\} \\
{\scriptstyle f}\uparrow \quad f(E) & \longmapsto & f_0(E_0) \quad \uparrow{\scriptstyle f_0} \\
\quad\ \ \uparrow & & \uparrow \quad\ \ \\
\quad\ \ E & \longmapsto & E_0 \quad\ \ \\
\{\text{Constructable subsets of } \operatorname{Spec} S\} & \longrightarrow & \{\text{Constructable subsets of } (\operatorname{Spec} S)_0\}
\end{array}$$

Where the horizontal maps are bijections. Note that f_0 is well-defined by the previous proposition, and the diagram commutes.

Def. If $\mathbb{k}=\bar{\mathbb{k}}$, and R, S are of finite type over $\mathbb{k}$.

$R \longrightarrow S \rightsquigarrow Y \triangleq \operatorname{Spec} S \longrightarrow \operatorname{Spec} R \triangleq X$

$$\begin{tikzcd} Y \arrow[d] & Y_0 \arrow[l] \arrow[d] \\ X & X_0 \arrow[l] \end{tikzcd}$$

Then X_0, Y_0 are the "$\mathbb{k}$-points" of X and Y, in the sense that:

$$\begin{tikzcd} R \arrow[r] & R/m \cong \mathbb{k} \\ \mathbb{k} \arrow[u] \arrow[ur, "\cong"] & \end{tikzcd}$$

$\Rightarrow$ we have a $\mathbb{k}$-algebra homomorphism $R \to \mathbb{k}$

$\Rightarrow \operatorname{Spec} \mathbb{k} \longrightarrow \operatorname{Spec} R$, the image is the closed point m.

So $X_0 \triangleq \operatorname{Hom}_{\mathbb{k}}(R, \mathbb{k}) \longleftarrow \operatorname{Hom}_{\mathbb{k}}(S, \mathbb{k}) \triangleq Y_0$

Example:

$X = \operatorname{Spec}(k[\{t_{ij}\}_{i,j=1}^n]/(T^2-T))$ $k=\bar{k}$. Here (T^2-T) stands for the ideal generated by the entries of T^2-T, $T=(t_{ij})$. Then X_0 is in 1-1 correspondence with the set of idempotent matrices.

Define: $GL_n \xrightarrow{\varphi_r} X$: $g \longmapsto g\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} g^{-1}$, where $GL_n = \operatorname{Spec}(k[\{g_{ij}\}_{i,j=1}^n, \frac{1}{\det g}])$ and $\operatorname{rank}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} = r$. Then φ_r corresponds to the ring map:

$$k[t_{ij}]/(T^2-T) \longrightarrow k[g_{ij}, \tfrac{1}{\det g}]$$

$$t_{ij} \longmapsto \text{the } (i,j)\text{-th entry of } g\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} g^{-1}$$

From Jordan canonical form we know that every idempotent matrix can be written in the form $g\,\operatorname{diag}(I_r, O_{n-r})g^{-1}$. Thus the image of φ_r in X_0 consists of all idempotents of $M_n(k)$. Since GL_n is irreducible, its image is also irreducible. When r ranges from 0 to n, the image ranges over all X. Moreover we see that we can cook up "functions" taking different values on different components $\varphi_r(GL_n)$, $r=0,\cdots,n$, namely $f \in \mathbb{k}[t_{ij}]/(T^2-T)$ $f([t_{ij}]) = \Sigma\, t_{ii}$ (Assume $\operatorname{char} k = 0$), and f takes the value r on $\varphi_r(GL_n) \Rightarrow f - r \in \bigcap_{m \in (\operatorname{Im}\varphi_r)_0} m = \mathfrak{p}_r \triangleq$ the minimal prime corresponding to the component $\varphi_r(GL_n)$. Hence we can see that X is disconnected.

(In $\operatorname{char} k > 0$, this f need not seperate all components, for instance, $\operatorname{char} k = 3$ $\operatorname{tr}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 1+1+1 = 0$ and $\operatorname{tr}\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} = 0$. However, we may cook up more sophisticated versions of "tr", namely, we may take the trace of $\Lambda^3\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ on $\Lambda^3 k^3 = k$, which is 1 on $\varphi_3(GL_3)$ but 0 on any other components $\varphi_i(GL_3)$ $i=0,1,2$)

§5. Artinian Rings.

- Nakayama's Lemma

Lemma: (Nakayama) R: a ring, M: an R-module.

(a). If M is finite, $IM = M$, then $\exists x \in I$ s.t. $(1+x)M = 0$

(b). If M is finite, $IM = M$ and $I \subseteq Rad(R)$ ($Rad R = \bigcap$ all maximal ideals of R), then $M = 0$

Pf: (a). Pick generators $m_1, \cdots, m_n$ of M. Then $M = IM \Rightarrow m_i = \sum f_{ij} m_j$, $f_{ij} \in I$
$\Rightarrow A \triangleq \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} - (f_{ij})$ annihilates the collumn vector $(m_1, \cdots, m_n)^t$.
$\Rightarrow 0 = A^* A \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \det A & & \\ & \ddots & \\ & & \det \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \det A \cdot m_1 \\ \vdots \\ \det A \cdot m_n \end{pmatrix} \Rightarrow \det A$ annihilates M.

But $\det A = \det(I + (f_{ij})) = 1 +$ polynomials in f_{ij} $= 1 + x$, $x \in I$.

(b) It suffices to show that $1+x$ is invertible. Suppose not, then $1 + x \in \mathfrak{m}$ for some maximal ideal, but $x \in Rad(R) \subseteq \mathfrak{m} \Rightarrow 1 \in \mathfrak{m}$, which is absurd. $\square$

Rmk: Typically, Nakayama's lemma is applied where R is local and I is the maximal ideal of R. Then if M is finite, any generator of the finite dimensional $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$ lifts to generators of M. ⌜Pf: Consider the submodule of M generated by the lifts, say, $m_1, \cdots, m_n$, then $\frac{M}{\mathfrak{m}M + \langle m_1, \cdots, m_n \rangle} = 0 \Rightarrow \mathfrak{m} \frac{M}{\langle m_1, \cdots, m_n \rangle} = \frac{\mathfrak{m}M + \langle m_1, \cdots, m_n \rangle}{\langle m_1, \cdots, m_n \rangle} = \frac{M}{\langle m_1, \cdots, m_n \rangle} \Rightarrow \frac{M}{\langle m_1, \cdots, m_n \rangle} = 0$. ⌟ People tend to make mistakes by forgetting to check M finitely generated.

As an application of Nakayama's lemma, we prove:

Lemma:

Let R be a Noetherian local ring. M finite module over R. $I \subseteq R$ is a proper ideal, then $\bigcap_{m \geq 0} I^m M = 0$. In particular, if we take $R = M = \Bbbk[x_1, \cdots, x_n]_{(x_1, \cdots, x_n)}$, $I = \mathfrak{m} = (x_1, \cdots, x_n)R$ the maximal ideal, then $\bigcap_{n \geq 0} \mathfrak{m}^n = 0$.

Pf: Take $N = \bigcap_{m \geq 0} I^m M$. By assumption, N is finitely generated.

By Artin-Rees lemma, $\exists c \in \mathbb{N}$, s.t. $I^n M \cap N = I^{n-c}(I^c M \cap N)$. But in our case, $I^k M \cap N = N$, $\forall k$. Hence if $n = c+1$, $N = IN$.

Nakayama $\Rightarrow N = 0$. $\square$

- Length of an R-module.

Def: R is a ring and M an R-module. The length of M as an R-module is defined as: $\text{length}_R M \triangleq \sup\{n \mid \exists \text{ filtration: } M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M\}$

Example:

$R = k[x]$, $M_1 = R$, then $\text{length}_R(M_1) = +\infty$, since in this case, $\forall n \in \mathbb{N}$ we may always find filtrations of length n: e.g. $(x^n) \subsetneq (x^{n-1}) \subsetneq \cdots \subsetneq (x) \subsetneq M_1$.

On the other hand, $M_2 = k[x]/(x^5)$ is a module of length 5: since the only maximal filtration is given by $k[x]/(x^5) \supsetneq (x)/(x^5) \supsetneq \cdots \supsetneq (x^4)/(x^5) \supsetneq 0$, any two subsequent quotients $\cong k$. the simple finite $k[x]$-module. $\Rightarrow \text{length}_R(M_2) = 5$.

Fact: Equivalence of category

$\{\text{Finite length } k[x]\text{-modules}\} \Longleftrightarrow \left\{\begin{matrix}\text{Pairs } (V, T) \text{ where } V \text{ is a finite dimensional vector} \\ \text{space with a linear operator acting on it}\end{matrix}\right\}$

Lemma:

$0 \to M' \to M \xrightarrow{\varphi} M'' \to 0$: a short exact sequence of R-modules. Then
$$\text{length}_R(M) = \text{length}_R(M') + \text{length}_R(M'')$$

Pf: Given filtrations for M', M'' respectively of length n', n'' (may be ∞) say: $M' \supsetneq M'_1 \supsetneq \cdots \supsetneq M'_{n'}$, $M'' \supsetneq M''_1 \supsetneq \cdots \supsetneq M''_{n''}$. Then the filtration on M given by $M \supsetneq \varphi^{-1}(M''_1) \supsetneq \cdots \supsetneq \varphi^{-1}M''_{n''} \supseteq M' \supsetneq M'_1 \supsetneq \cdots \supsetneq M'_{n'}$ is of length $\geq n' + n''$.

Conversely, suppose $M_0 \subsetneq \cdots \subsetneq M_n = M$ is a filtration of M of length n. Consider $M'_n = M_i \cap M'$, $M''_n = \varphi(M_i)$. Let $n' = \#\{\text{strict inclusions in } M'_n\}$ and $n'' = \#\{\text{strict inclusions in } M''_n\}$.

Claim: if $M'_i = M'_{i+1}$, $M''_i = M''_{i+1}$, then $M_i = M_{i+1}$.

Indeed, we have by definition the exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & M'_i \arrow[r] \arrow[d, "\iota'"] & M_i \arrow[r] \arrow[d, "\iota"] & M''_i \arrow[r] \arrow[d, "\iota''"] & 0 \\ 0 \arrow[r] & M'_{i+1} \arrow[r] & M_{i+1} \arrow[r] & M''_{i+1} \arrow[r] & 0 \end{tikzcd}$$

Now if ι', ι'' are identities, by five lemma, so is ι.

Hence in our setting, either $M'_i \neq M'_{i+1}$ or $M''_i \neq M''_{i+1} \Rightarrow n' + n'' \geq \text{length}_R M$. $\square$

Lemma:

Suppose R is a Noetherian local ring with maximal ideal m. M a finite R-module. If $m^n M \neq 0$, $\forall n$, then $\text{length}_R(M)$ ($\triangleq \ell_R(M)$ for short) $= +\infty$.

Pf: By Nakayama's lemma, all the inclusions in the filtration: $M \supseteq mM \supseteq m^2 M \supseteq \cdots$ are strict. (otherwise $m^k M = m^{k+1} M \Rightarrow m \cdot m^k M = m^{k+1} M = m^k M \Rightarrow m^k M = 0$). $\square$

On the other hand, we have:

Lemma:

R: local Noetherian, $m \subseteq R$ its maximal ideal. M an R-module and $mM = 0$. Then $\ell_R(M) = \dim_{R/m}(M)$.

Pf: Note that M is a well-defined R/m-vector space. Any submodule is just a subspace. $\square$

Lemma:

R: Noetherian local ring, M: a finite R module with $m^n M = 0$ for some n. Then $\ell_R(M) < \infty$.

Pf: Consider the filtration, $M \supseteq mM \supseteq \cdots \supseteq m^n M = 0$. If each step is of finite length, then $\ell_R(M) = \sum_{i=0}^{n-1} \ell_R(m^i M / m^{i+1} M) < \infty$, by additivity of length. But each $m^i M / m^{i+1} M$ is finte and killed by m, and thus is of finite length. $\square$

- "Mathematics is a sequence of tautology..." — de Jong

- Artinian Rings

Def: A ring is called Artinian if the descending chain condition holds for ideals. ($\Rightarrow$ a.c.c. for closed subsets of $\operatorname{Spec} R$)

Examples:

1). Finite rings (e.g. $\mathbb{Z}/n$) are Artinian.

2). Given a field k, all finite dimensional k-algebras are Artinian. In fact, if R is of finite type over k, then R is Artinian $\Leftrightarrow \dim_k R < \infty$.

Lemma:

R: Artinian $\Rightarrow$ R has a finite number of maximal ideals.

Pf: If $\{m_1, m_2, \cdots\}$ were an infinite sequence of distinct maximal ideals. Then we obtain an infinitely ascending chain of closed sets of $\operatorname{Spec} R$, namely:

$Z_i = \{m_1, \cdots, m_i\} \subseteq \operatorname{Spec} R$. $\quad Z_1 \subsetneq Z_2 \subsetneq Z_3 \subsetneq \cdots$.

(Algebraically: if there were infinitely many distinct maximal ideals, we may cook up a descending chain of ideals:

$$R \supseteq m_1 \supseteq m_1 \cap m_2 \supseteq m_1 \cap m_2 \cap m_3 \supseteq \cdots$$

The inclusions are strict, by Chinese remaider theorem: $\frac{R}{m_1 \cap \cdots \cap m_k} \cong \frac{R}{m_1} \oplus \cdots \oplus \frac{R}{m_k}$.)

Lemma:

If R is Artinian $\Rightarrow$ $\operatorname{Rad}(R)$ is nilpotent.

Pf: Set $I = \operatorname{Rad} R$, and look at $I \supseteq I^2 \supseteq \cdots$

R Artinian $\Rightarrow I^n = I^{n+1} = \cdots$ Set $J = \{x \mid x I^n = 0\} \subseteq R$.

To get a contradiction, assume that $J \neq R$. Pick $J \subsetneq J'$, J' a minimal element among ideals strictly containing J. Pick $x \in J'$ $x \notin J$, then by minimality, $J' = (x) + J$. $\Rightarrow J'/J$ is finite. Nakayama $\Rightarrow I\, J'/J \subsetneq J'/J \Rightarrow IJ' \subsetneq J' \Rightarrow$ $IJ' \subseteq J$ by minimality again. $\Rightarrow J' I \cdot I^n \subseteq J I^n = 0$, but $J' I \cdot I^n = J' I^{n+1} = J' I^n$ $\Rightarrow J' \subseteq J$, contradiction. $\square$

Lemma:

R a ring. $I \subseteq R$ a nilpotent ideal. Then idempotents of $R \overset{1:1}{\Longleftrightarrow}$ idempotents of R/I.

Pf: (Geometric):

$$\begin{tikzcd}
\{\text{Idempotents of } R\} \arrow[r, leftrightarrow, dashed, "\exists 1:1 \text{ by}", "\text{commuting diagram}"'] \arrow[d, leftrightarrow, "1:1"] & \{\text{Idempotents of } R/I\} \arrow[d, leftrightarrow, "1:1"] \\
\{\text{Connected components of } \operatorname{Spec} R\} \arrow[r, leftrightarrow, "1:1"] & \{\text{Connected components of } \operatorname{Spec} R/I\}
\end{tikzcd}$$

(Algebraic): this method (Newton) also works for non-commutative rings.

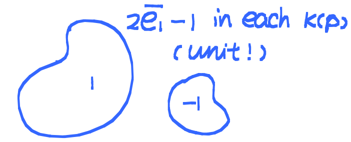
For any idempotent $\bar{e}$ of R/I, we need to find an idempotent e of R mapping to $\bar{e}$.

First take any preimage of $\bar{e}$ in R, say e_1, then

$$e_1^2 - e_1 = x \in I$$

Choosing different lifts $e_2 = e_1 + y$ gives

$$\begin{aligned} e_2^2 - e_2 &= e_1^2 + 2e_1 y + y^2 - e_1 - y \\ &= (2e_1 - 1)y + y^2 + x \overset{?}{\equiv} 0 \mod I^2 \end{aligned}$$



Indeed the equation can be solved since $2e_1 - 1$ is a unit: $(2e_1-1)^2 = 4(e_1^2 - e_1) + 1 = 1 + x$ and x is nilpotent. Thus we may just take $y = \frac{x}{1-2e_1} \in I$, and the difference of $e_2^2 - e_2$ is in I^2. Inductively this can be done mod I^n for all n, but since I is nilpotent, $I^n = 0$ for some n, and $e_n^2 - e_n = 0$. $\square$

Upshot: An Artinian ring is the product of its localizations at maximal ideals and the only primes are the maximals.
($R = \mathbb{Q} \times \mathbb{Q}$, localizations give copies of $\mathbb{Q}$, as well as taking quotients)

Lemma:

Any ring with finitely many maximal ideals and whose radical is nilpotent (locally nilpotent) is the product of its localizations at maximal ideals. Moreover, any prime is maximal.

Pf: Let $m_1, \cdots, m_n$ be the maximal ideals of R. $\mathrm{Rad}(R) = \bigcap m_i$. Nilpotency of $\mathrm{Rad}(R) \Rightarrow \mathrm{Rad}(R) \subseteq \bigcap_{\text{primes}} p \subseteq p$, for any prime p of R. $\Rightarrow p \supseteq \bigcap m_i \Rightarrow p \supseteq m_i$ for some i. $\Rightarrow p = m_i$, thus maximal.

By Chinese remainder thm, we have $R/\mathrm{Rad}(R) \cong R/m_1 \times \cdots \times R/m_n$. By the previous lemma, we can find idempotents $\bar{e}_1, \cdots, \bar{e}_n$ of $R/\mathrm{Rad}(R)$ corresponding to $(0, \cdots, 1, \cdots 0)$ in $R/m_1 \times \cdots \times R/m_n$ ($\bar{e}_i \bar{e}_j = \delta_{ij} \bar{e}_i$), whose lifts in R give rise to the ring decomposition $R \cong Re_1 \times \cdots \times Re_n$, which is a decomposition of R into local rings. ($e_i \notin m_i$ but $e_i \in m_j$, $\forall j \neq i$; $Re_i \cong Re_i$ which is local with maximal ideal m_i ; $\mathrm{Spec}(Re_i) = \{m_i\}$). $\square$

Lemma:

A ring R is Artinian iff it is a module of finite length over itself. If so, R

is Artinian and Noetherian, and is equal to products of localizations at maximal ideals.

Pf: $l_R(R) < \infty \Leftrightarrow$ both a.c.c. and d.c.c hold for R

$\Leftrightarrow$ R is Noetherian and Artinian.

It suffices to show that, R Artinian $\Rightarrow l_R(R) < +\infty$.

Now R Artinian $\Rightarrow R = R_1 \times \cdots \times R_m$, with R_i local Artinian. Moreover, by $0 \longrightarrow R_i \longrightarrow R_1 \times \cdots \times R_i \longrightarrow R_1 \times \cdots \times R_{i-1} \longrightarrow 0$ and additivity of the length function it suffices to prove for the Artinian local case.

Now, $R \supseteq m \supseteq \cdots \supseteq m^n \supseteq m^{n+1} = 0$. $\Rightarrow l_R(R) = \sum_{i=0}^{n} l_R(\frac{m^i}{m^{i+1}}) = \sum_{i=0}^{n} \dim_{R/m}(\frac{m^i}{m^{i+1}})$.

Moreover, each m^i as R-module also satisfies d.c.c. $\Rightarrow \dim_{R/m} m^i/m^{i+1} < \infty$. $\square$

§6. K-Groups

R: a ring. We define two functors: $K_0'(R)$, $K_0(R)$: $\mathcal{M}od_R \longrightarrow \mathcal{A}b$.

Def: ($K_0'(R)$) For every finitely generated R-module M, $\exists$ an element $[M] \in K_0'(R)$ and for every short exact sequence $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of finite R-modules $[M] = [M'] + [M'']$ in $K_0'(R)$. Furthermore, it's the free object with these two properties.

Def: ($K_0(R)$) For every finite projective R-module M, $\exists$ an element $[M] \in K_0'(R)$, and for every short exact sequence $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of finite prj. R-modules $[M] = [M'] + [M'']$ in $K_0'(R)$. Furthermore, it's the free object with these two properties.

(About freeness: What you see is what you get! — de Jong)

Lemma:

Let R be an Artinian ring. Then $\text{length}_R(\cdot)$ induces an abelian group homomorphism $K_0'(R) \longrightarrow \mathbb{Z}$, $[M] \longmapsto \text{length}_R(M)$.

Pf: Any $M \cong R^n/N$ is of finite length, since R is. Furthermore length_R is additive. $\square$

Note that there is a canonical homomorphism: $K_0(R) \longrightarrow K_0'(R)$

Examples:

(1). $R = k$ a field, then $K_0(R) \cong K_0'(R) \xrightarrow[\sim]{\dim} \mathbb{Z}$.

(2). Let k be a field. $K_0(k[x]) \cong K_0'(k[x]) \xrightarrow[\sim]{\text{rank}} \mathbb{Z}$.

Indeed, by the structure thm of finitely generated modules over a principal ideal domain, any M finitely generated, then $M \cong R^n \oplus \bigoplus_i \frac{R}{(f_i^{n_i})}$, f_i irreducible.

Hence M projective $\Leftrightarrow$ M free. $K_0(R) \xrightarrow{\sim} \mathbb{Z}$ trivially.

Furthermore, we can show that: if N torsion ($N \cong R/(f_i^{n_i})$) then $[N] = 0$ in $K_0'(R)$. Indeed, $0 \to R \xrightarrow{f_i^{n_i}} R \longrightarrow N \longrightarrow 0 \Rightarrow [R] = [R] + [N] \Rightarrow [N] = 0$.

Now rank: $K_0'(R)$ is well-defined since rank is additive.

Now if $\text{rank}(\Sigma n_i [M_i]) = 0 \Rightarrow \Sigma n_i \,\text{rank} M_i = 0$. Group together those n_i which are positive or negative, we obtain: $\sum_{n_i > 0} n_i \,\text{rank} M_i = k = \sum_{n_j < 0} n_j \,\text{rank} M_j$

$$0 \to R^k \oplus R^{k'} \to R^k \oplus R^{k'} \to \oplus M^{|n_i|} \to 0 \quad (n_i>0 \text{ or } n_i<0)$$

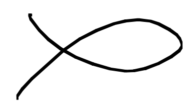
$$\Rightarrow \sum_{n_i>0} n_i[M_i] - \sum_{n_j<0} |n_j|[M_j] = \sum_{n_i>0}[R^{k+k'}] - \sum_{n_i>0}[R^{k+k'}] + \sum_{n_j<0}[R^{k+k''}] - \sum_{n_j<0}[R^{k+k''}] = 0.$$

(3). Let k be a field, $R = \{f \in k[x] \mid f(0)=f(1)\}$. In this case $K_0(R) \cong k^* \oplus \mathbb{Z}$
$K_0'(R) \cong \mathbb{Z}$.

$R \cong k[A,B]/(A^3-B^2+AB)$

$$\frac{k[A,u]}{(A-u^2+u)} \leftarrow \frac{k[A,B]}{(A^3-B^2+AB)} \qquad B \mapsto Au \qquad A^3 - A^2u^2 + A^2u = A^2(A-u^2+u)$$

$$\cong k[u] \Rightarrow \quad k[u]e_1 \oplus \cdots \oplus k[u]e_n$$



$$\frac{k[B,s]}{(-1+s+B)} \qquad k[s]e_1' \oplus \cdots \oplus k[s]e_n'$$

$$e_1 \mapsto (\lambda + f(u))e_1'$$

$B^3s^3 - B^2 + B^2s$ $\qquad = \lambda + \frac{1}{s^k} f'(s)$

$B^2(Bs^3 - 1 + s)$ $\qquad s(1+Bs^2) = 1$

Fact: $K_0'(R_1\times R_2)\cong K_0'(R_1)\times K_0'(R_2)$; $K_0(R_1\times R_2)\cong K_0(R_1)\times K_0(R_2)$.

Lemma:
Suppose R is local, $K_0(R)\xrightarrow[\sim]{\text{rank}}\mathbb{Z}$.
Pf: First of all we can show that if R is local, then M projective $\Rightarrow M$ free. Indeed, let $m_1,\cdots,m_n$ be a minimal set of generators $\Rightarrow 0\to\ker\varphi\to R^n\xrightarrow{\varphi}M\to 0$ splits since M projective. $\Rightarrow R^n=M\oplus\ker\varphi \Rightarrow \otimes K$: $K^n=\frac{M}{mM}\oplus\frac{\ker\varphi}{m\ker\varphi}$. By Nakayama's lemma $M/mM\cong K^n\Rightarrow \ker\varphi/m\ker\varphi=0\Rightarrow\ker\varphi=0$, by Nakayama again. (note that $\ker\varphi$ is necessarily finitely generated, since $R^n\to\ker\varphi$ is also surjective).

Thus rank: $K_0(R)\to\mathbb{Z}$ is well-defined, since $0\to M'\to M\to M''\to 0$ s.e.s of projective modules $\Rightarrow M'\cong R^n$, $M''\cong R^m$, $M\cong R^{n+m}$ for some $n,m\in\mathbb{Z}$. rank$(\Sigma a_i[R^{n_i}])=0 \Leftrightarrow 0\to\bigoplus_{a_i>0}R^{a_in_i}\to\bigoplus_{a_i<0}R^{|a_i|n_i}\to 0$ exact. $\square$

Lemma:
Suppose R is Artinian local, then we have a commutative diagram:

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\text{rank}"', "\cong"] & K_0'(R) \arrow[d, "\cong"', "\text{length}"] \\ \mathbb{Z} \arrow[r, "\cdot\,\text{length}_R(R)"] & \mathbb{Z} \end{tikzcd}$$

Pf: Since the maps are well-defined, it suffices to check for generators of $K_0(R)$, namely free modules over R, which again reduces to check for R itself: $\text{rank}_R(R)\cdot\text{length}_R(R)=\text{length}_R(R)$. $\square$

§7. Graded Rings and Modules.

Def. (Graded Ring) $S = \oplus_{d\geq 0} S_d$. $S_d \cdot S_e \subseteq S_{d+e}$. $S_+ = \oplus_{d\geq 1} S_d$ is an ideal of S, called the irrelevent ideal.

Def. (Graded Module) $M = \oplus_{d\in \mathbb{Z}} M_d$, $S_d \cdot M_e \subseteq M_{d+e}$.

Lemma.

S is Noetherian $\Leftrightarrow$ S_0 is Noetherian and S_+ is finitely generated as an S-module. Furthermore, homogeneous elements $f_1, \cdots, f_n \in S_+$ generate S as an S_0 algebra iff they generate S_+ as an ideal.

Lemma.

If M is a finitely generated graded S-module and S is noetherian, then each M_d is a finite S_0-module.

Def. (Numerical Polynomial) A: an abelian group. A function $f: \mathbb{Z} \longrightarrow A$ defined for all but finitely many integers is called a numerical polynomial if $\exists\, r\geq 0$, $a_0, \cdots, a_r \in A$ s.t. $f(n) = \sum_{i=0}^{n} \binom{n}{i} a_i$

Examples

(a). $S = k[x_1, \cdots, x_n]$. $\deg x_i = 1$. $\dim(k[x_1, \cdots, x_n]_d) = \binom{n+d-1}{d-1} \triangleq f(d)$ is a numerical polynomial.

(b). $S = k[x]$. $\deg x = 2$. $\dim k[x]_d = \begin{cases} 0 & d: \text{odd} \\ 1 & d: \text{even} \end{cases}$ can't be numerical.

Lemma.

Suppose $f: n \mapsto f(n) \in A$ defined for $n \gg 0$ satisfies $n \mapsto f(n) - f(n-1)$ is numerical, then f is numerical.

Pf: Write $f(n) - f(n-1) = \sum_{i=0}^{r} \binom{n}{i} a_i$. Then $g(n) \triangleq f(n) - \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ satisfies

$$g(n) - g(n-1) = f(n) - f(n-1) - \sum_{i=0}^{r} \left[\binom{n+1}{i+1} - \binom{n}{i+1}\right] a_i$$
$$= f(n) - f(n-1) - \sum_{i=0}^{r} \binom{n}{i} a_i$$
$$= 0 \quad \text{for } n \gg 0$$

$\Rightarrow g(n) \equiv a_{-1} \in A$ is a constant

$\Rightarrow f(n) = a_{-1} + \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ for $n >> 0$. $\square$

Now given S: Noetherian graded ring. M: finite, graded S-module over S.
Define $\Phi_M : d \mapsto [M_d] \in K_0'(S_0)$

Thm. Assume S_+ generated by elements in degree 1. Then the above function $\mathbb{N} \longrightarrow K_0'(S_0)$ is a numerical polynomial.

Pf: Let M be generated over $m_1, \cdots, m_s$, homogeneous elements of degree $d_1, \cdots, d_s$. Now we prove by induction on the number of generators r of S_+ over S_0.

$r = 0$. $S = S_0$. Then $M_d = 0$ for $d > \max\{d_1, \cdots, d_s\}$

$r > 0$, let $x \in S_1$ be one of a minimal set of generators, so that we can apply the induction hypothesis to $S/xS \cong S_0 \oplus \frac{S_1}{(x)} \oplus \frac{S_2}{(x)} \oplus \cdots$

Case I: x is nilpotent on M. This can be proved by induction on e: $x^e M = 0$ (M is killed by x^e by Noetherian hypothesis)

$e = 1$. M is a module over S/xS. The above induction hypothesis applies.

Suppose it's true for $< e$. For e, we can find a short exact sequence of graded modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ s.t. M', M'' are killed by $x^{e'}$ $x^{e''}$, $e', e'' < e$, then the induction hypothesis of this case applies. Such a s.e.s. can be taken, for instance, to be $M' = xM$, $M'' = M/xM'$. Then for each degree, we have $0 \longrightarrow M'_d \longrightarrow M_d \longrightarrow M''_d \longrightarrow 0$, $\forall d \in \mathbb{Z}$

$\Rightarrow \Phi_M(d) = \Phi_{M'}(d) + \Phi_{M''}(d)$

$\Rightarrow \Phi_M$ is a numerical polynomial since $\Phi_{M'}$ and $\Phi_{M''}$ are, by induction.

Case II. x is not nilpotent on M. Let $M' = \{m \in M \mid x^t m = 0$ for some $t \in \mathbb{N}\}$
Then M' is a finitely generated submodule of M by Noetherian hypothesis, we may then form $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, where $M'' \cong M/M'$, and x acts on M'' as a non-zero divisor. Since $\Phi_M = \Phi_{M'} + \Phi_{M''}$, it reduces to prove for M'' on which x is a non-zero divisor.

Case III. x is a non-zero divisor on M. Set $\overline{M}=M/xM$. Then. $\forall d$ $0\longrightarrow M_{d-1}\xrightarrow{x} M_d\longrightarrow \overline{M}_d\longrightarrow 0$ is short exact and $\overline{M}$ is an S/xS module so that the induction hypothesis may be applied and $\Phi_{\overline{M}}$ is a numerical polynomial. $\Rightarrow$ $\Phi_M(d)-\Phi_M(d-1)=\Phi_{\overline{M}}(d)$. Then $\Phi_M(d)$ is a numerical polynomial by our previous lemma. $\square$

Remark: Morphisms of graded modules $\varphi: M\longrightarrow N$ is an S-map and $\varphi(M_d)\subseteq N_d$. Thus multiplication by homogeneous elements of S are not necessarily morphisms of graded modules. Yet the quotient module M/xM is still graded and $M\longrightarrow M/xM$ is a graded morphism.

Examples

(1). $R=M=k[x_0,\cdots,x_n] \Rightarrow R_0=k$, $K_0'(R_0)\overset{\dim}{\simeq}\mathbb{Z}$, $\dim_k M_d=\binom{d+n}{d}$

(2). $R=M=R/(f)$ f: homogeneous of deg d. $\dim_k M_m=\binom{m+n}{m}-\binom{m-d+n}{m-d}$.

• Application: Noetherian Local Rings

R: Noetherian local, with maximal ideal m, residue field $\kappa=R/m$.

M: a finite R-module.

Def: (Hilbert function of M over R) $\varphi_M: \mathbb{N}\longrightarrow\mathbb{N}$. $n\longmapsto \mathrm{length}_R(m^nM/m^{n+1}M)$ ($=\dim_\kappa(m^nM/m^{n+1}M)$, by Nakayama's lemma).

Note that $\mathrm{length}_R(M/m^{n+1}M)=\sum_{i=0}^{n}\varphi_M(i)$.

Def: (a variant) Let $I\subseteq R$ be an ideal of definition, i.e. $\sqrt{I}=m$ ($\Leftrightarrow m^n\subseteq I$ for some $n\in\mathbb{N}$. in particular, we see that $M/m^{nk}M\twoheadrightarrow M/I^kM$. and thus M/I^nM has finite length). $\varphi_{M,I}(n)\triangleq \mathrm{length}_R(I^nM/I^{n+1}M)$

Prop: $\varphi_{M,I}$ is a numerical polynomial.

Pf: Consider the graded ring $S=R/I\oplus I/I^2\oplus I^2/I^3\oplus\cdots$, and graded module $N=M/IM\oplus IM/I^2M\oplus I^2M/I^3M$. Then S is a graded ring generated by degree

1 elements and M is a finite S-module, and thus the thm applies. Here we identify $K'(S_0) = K'(R/I) \cong \mathbb{Z}$ by taking length, since R/I is Artinian. $\square$

Lemma:

Suppose $0 \to M' \to M \to M'' \to 0$ is a s.e.s. of finite R-modules. Then there exists a submodule $N \subseteq M'$ of finite colength (i.e. M'/N of finite length) and an integer $c \in \mathbb{N}$ s.t. $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n)$, $n >> 0$.

Pf: We have a surjective map $M/I^nM \xrightarrow{\varphi} M''/I^nM'' \to 0$, and $\ker\varphi = \frac{M'}{M' \cap I^nM}$.

$\Rightarrow 0 \to \frac{M'}{M' \cap I^nM} \to \frac{M}{I^nM} \to \frac{M''}{I^nM''} \to 0$ is a s.e.s.

Moreover, by Artin-Rees lemma $I^nM \cap M' = I^{n-c}(I^cM \cap M')$ for all $n \geq c$. Let $N = I^cM \cap M'$, then N is of finite colength ($\frac{M'}{N} \cong \frac{M'+I^cM}{I^cM} \subseteq \frac{M}{I^cM}$)

$\Rightarrow 0 \to \frac{M'}{I^{n-c}N} \to \frac{M}{I^nM} \to \frac{M''}{I^nM''} \to 0$ s.e for $n \geq c$.

Take length, and we obtain: $\ell_R(\frac{M}{I^nM}) = \ell_R(\frac{M'}{I^{n-c}N}) + \ell_R(\frac{M''}{I^nM''}) = \ell_R(\frac{M}{N}) + \ell_R(\frac{N}{I^{n-c}N}) + \ell_R(\frac{M''}{I^nM''})$

$\Rightarrow \sum_{i=0}^{n-1} \varphi_{I,M}(i) = \ell_R(\frac{M}{N}) + \sum_{i=0}^{n-1} \varphi_{I,N}(i) + \sum_{i=0}^{n-1} \varphi_{I,M''}(i)$ for $n \geq c$.

Taking differences of neighboring n, $(n+1)$ th equation

$\Rightarrow \varphi_{I,M}(n) = \varphi_{I,N}(n) + \varphi_{I,M''}(n)$. $\square$

Lemma:

If $M' \subseteq M$ is of finite colength, then $\exists C_1, C_2 \in \mathbb{N}$ s.t. :

$\varphi_{I,M}(n) \geq \varphi_{I,M'}(n-C_1) - C_2$ and $\varphi_{I,M'}(n) \geq \varphi_{I,M}(n) - C_2$

Pf: Consider the s.e.s: $0 \to \frac{M'}{M' \cap I^nM} \to \frac{M}{I^nM} \to \frac{M}{M'+I^nM} \to 0$. $M' \subseteq M$ of finite colength

$\Rightarrow I^{c_1}M \subseteq M'$ for some c_1 $\Rightarrow$ For $n \geq c_1$ $0 \to \frac{M'}{I^nM} \to \frac{M}{I^nM} \to \frac{M}{M'} \to 0$ is exact.

But $I^nM = I^{n-c_1} \cdot I^{c_1}M \subseteq I^{n-c_1}M' \Rightarrow \ell_R(\frac{M'}{I^nM}) \geq \ell_R(\frac{M'}{I^{n-c_1}M'})$. Hence

$$\ell_R(\frac{M}{I^nM}) = \ell_R(\frac{M'}{I^nM}) + \ell_R(\frac{M}{M'}) \geq \ell_R(\frac{M'}{I^{n-c_1}M'}) + \ell_R(\frac{M}{M'})$$

$\Rightarrow \varphi_{I,M}(n) \geq \varphi_{I,M'}(n-c_1)$ for $n >> 0$

Moreover $\frac{M'}{I^nM'} \twoheadrightarrow \frac{M'}{I^nM} \Rightarrow \ell_R(\frac{M'}{I^nM'}) \geq \ell_R(\frac{M'}{I^nM}) = \ell_R(\frac{M}{I^nM}) - \ell_R(\frac{M}{M'})$ by the s.e.s above.

$\Rightarrow \varphi_{I,M'}(n) \geq \varphi_{I,M}(n)$ for $n >> 0$.

Now take C_2 large, then $\varphi_{I,M}(n) \geq \varphi_{I,M'}(n-C_1) - C_2$, $\varphi_{I,M'}(n) \geq \varphi_{I,M}(n) - C_2$ for all $n \in \mathbb{N}$. $\square$

Lemma:

Suppose I, I' are ideals of definition, and M a finite R-module. Then there exists an $a \in \mathbb{N}$ s.t. $\sum_{i=0}^{n} \varphi_{I,M}(i) \leq \sum_{i=0}^{an} \varphi_{I',M}(i)$.

Pf: Indeed, since $I'^{a-1} \subseteq I$ for some $a \in \mathbb{N}$.

$\Rightarrow \sum_{i=0}^{n} \varphi_{I,M}(i) = \ell_R(M/I^{n+1}M) \leq \ell_R(M/I'^{(a-1)(n+1)}M) \leq \ell_R(M/I'^{an+1}) = \sum_{i=0}^{an} \varphi_{I',M}(i)$.

(since $(a-1)(n+1) \leq an+1$ for $n \geq a-1$) $\square$

Def. R a local Noetherian ring, M finite over R. $\chi_{I,M}(n) \triangleq \text{length}_R(M/I^{n+1}M)$.

Then $d(M) \triangleq \deg \chi_{I,M}$

$$= \begin{cases} 0 & \ell_R(M) < \infty \\ \deg(\varphi_{I,M}) + 1 & \text{otherwise} \end{cases}$$

This definition of $d(M)$ is independent of choices of ideals of definition by the previous lemma: $\chi_{I,M}(n) \leq \chi_{I',M}(an) \leq \chi_I(a'an) \Rightarrow \deg \chi_{I,M} = \deg \chi_{I'M}$.

Remark: Why the flexibility about choosing I?

Example: Consider $R = k[x,y,z]/(z^2 - xy)$ $m = (x,y,z)$. Then

Lemma:

If $M' \hookrightarrow M$ is of finite colength, but neither has finite length, then:

$\chi_{I,M} - \chi_{I,M'}$ has degree strictly less than the degree of either $\chi_{I,M}$ or $\chi_{I,M'}$.

Pf: By a previous lemma, $\exists\, C_1, C_2$, s.t.

$$\begin{cases} \varphi_{I,M}(n) \geq \varphi_{I,M'}(n - C_1) - C_2 \\ \varphi_{I,M'}(n) \geq \varphi_{I,M}(n) - C_2 \end{cases}$$

Moreover, $\deg \varphi_{I,M}(n) = d \geq 0 \Rightarrow \frac{1}{n^d}(\varphi_{I,M'}(n) + C_2) \geq \frac{1}{n^d} \varphi_{I,M'}(n) \geq \frac{1}{n^d}(\varphi_{I,M'}(n - C_1) - C_2)$

Letting $n \to \infty \Rightarrow$ the leading coefficient of $\varphi_{I,M}$ and $\varphi_{I,M'}$ are the same.

$\Rightarrow \chi_{I,M} - \chi_{I,M'}$ has 0-coefficient for n^{d+1}. $\square$

Lemma:

$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, s.e.s. Then:

$\max\{\deg \chi_{I,M'}, \deg \chi_{I,M}\} = \deg \chi_{I,M}$. Furthermore, if M' doesn't have finite length, then $\deg(\chi_{I,M} - \chi_{I,M'} - \chi_{I,M''}) < \deg(\chi_{I,M})$.

Pf: By one of the previous lemmas, $\varphi_{I,M}(n) = \varphi_{I,N}(n) + \varphi_{I,M}(n)$ for $n >> 0$ and $M' \hookrightarrow N$ is of finite colength. In particular, $\varphi_{I,N}$ and $\varphi_{I,M'}$ have the same leading coefficient and degree. $\Rightarrow$ $\chi_{I,M}(n)$, $\chi_{I,N}(n) + \chi_{I,M''}(n)$ (which has the same leading coefficient and degree as $\chi_{I,M'}(n) + \chi_{I,M''}(n)$, in any case) have the same leading coefficient and degree. The result follows. $\square$

§8. Dimension

Def: The Krull dimension of a ring R is the Krull dimension of R as a topological space, i.e. $\dim R = \sup\{n \mid \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,\ \mathfrak{p}_i \subseteq R \text{ prime}\}$

Def: The height of $\mathfrak{p} \subseteq R$ is defined as $\dim R_{\mathfrak{p}}$ (codim of $V(\mathfrak{p})$ in $\operatorname{Spec} R$)

E.g. $\dim R = 0$ iff every prime is maximal.

Lemma: $\dim R = \sup\{\text{height } \mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$. $\square$

Lemma:
R: Noetherian ring of dim 0 $\Leftrightarrow$ R: Artinian.
Pf: The converse is proved already.
Now, assume R is Noetherian of dim 0. $\Rightarrow$ $\operatorname{Spec} R$ is Noetherian $\Rightarrow$ $\operatorname{Spec} R = Z_1 \cup \cdots \cup Z_r$, a finite union of irreducible components. Now since $Z_i = Z(\mathfrak{p}_i)$ for minimal primes $\mathfrak{p}_i$'s and $\dim R = 0$, $\mathfrak{p}_i$'s are maximal. $\Rightarrow$ $\operatorname{Spec} R = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$ (every prime contains a minimal prime). Set $I = \bigcap \mathfrak{p}_i = \operatorname{rad} R \Rightarrow I^N = 0$ for some N since R is Noetherian. $\Rightarrow R = \prod R_i$, $R_i = R_{\mathfrak{p}_i}$. Now R_i is local, Noetherian with exactly one prime, and which is nilpotent (nilradicals localize!) $\Rightarrow \ell(R_i) < \infty \Rightarrow \ell(R) < \infty$. $\square$

Cor:
R: Noetherian local, then $\dim R = 0 \Leftrightarrow d(R) = 0$
Pf: $d(R) = 0 \Leftrightarrow \ell_R(R) < \infty$ $\square$

Upshot: R: Noetherian local. $\dim R = 0 \Leftrightarrow d(R) = 0 \Leftrightarrow \mathfrak{m} = \sqrt{0} \Leftrightarrow$ Artinian $\cdots$
How about $\dim R = 1$?

Lemma:
R: Noetherian local. TFAE:

1). $\dim R = 1$

2). $d(R) = 1$

3). $\exists x \in \mathfrak{m}$, x not nilpotent, s.t. $\mathfrak{m} = \sqrt{(x)}$

4). $\exists$ an ideal of definition generated by 1 element but $\nexists$ an ideal of definition generated by 0.

Pf: 3) $\Leftrightarrow$ 4) is easy.

1) $\Rightarrow$ 3). Assume $\dim R = 1$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be minimal primes of R. Then the only other prime is $\mathfrak{m}$. Since $\mathfrak{m} \not\supseteq \mathfrak{p}_i$, by prime avoidance, $\exists x \in \mathfrak{m}$. $x \notin \mathfrak{p}_i$, $\forall i$. Hence the only prime containing x is $\mathfrak{m}$ and x is not nilpotent. $\Rightarrow V(x) = \mathfrak{m}$ and $\sqrt{(x)} = \mathfrak{m}$

3) $\Rightarrow$ 2). Assume $\exists x \in \mathfrak{m}$, x not nilpotent, $\mathfrak{m} = \sqrt{(x)}$. Because x is not nilpotent $d(R) \neq 0$ (the upshot above). On the other hand, $I = (x)$ is an ideal of definition, so we may compute $d(R) = \deg(\varphi_{I,R}) + 1$. In other words, we now look at $\varphi_{I,R}(n) = \text{length}_R(\frac{I^n}{I^{n+1}}) = \ell_R(\frac{(x^n)}{(x^{n+1})})$. Since $\frac{R}{(x)} \to \frac{(x^n)}{(x^{n+1})} \to 0 \Rightarrow \ell(\frac{(x^n)}{(x^{n+1})}) < \ell(\frac{R}{(x)})$
$\Rightarrow \varphi_{I,R}(n) < \text{Const}$. $\forall n$. $\Rightarrow \deg \varphi_{I,R} = 0 \Rightarrow d(R) = 1$.

2) $\Rightarrow$ 1). Argue by contradiction. Suppose $\exists\ \mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$ distinct primes. Pick an ideal of definition, and consider $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$. By assumption, we have $d(R) = \max\{d(\mathfrak{p}), d(R/\mathfrak{p})\} = 1$ (Note that $d(R/\mathfrak{p})$ is the same regardless of viewing it as an R or $R/\mathfrak{p}$-module) $\Rightarrow d(R/\mathfrak{p}) \leq 1$. Clearly $d(R/\mathfrak{p}) \neq 0$ since $R/\mathfrak{p}$ is not 0-dimensional. Now, pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p} \Rightarrow 0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to \frac{R}{(x)+\mathfrak{p}} \to 0$ is exact. $\Rightarrow \chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(x)+\mathfrak{p}} = -\chi_{I,R/(x)+\mathfrak{p}}$ has degree < 1
$\Rightarrow d(R/(x)+\mathfrak{p}) = 0 \Rightarrow \dim R/(x)+\mathfrak{p} = 0$, which is absurd since $\mathfrak{q} \subsetneq \mathfrak{m}$. $\square$

Remark: In 2) $\Rightarrow$ 1) we used the fact that if $R \twoheadrightarrow R'$ and M' is an R'-module $\Rightarrow \ell_R(M) = \ell_{R'}(M)$, and if I is an ideal of definition in R, then IR' is an ideal of definition in R'.

More generally, we can show by doing dimension reduction to show that:

Lemma:

R: local Noetherian ring, $d \geq 0$. TFAE:

1). $\dim R = d$

2). $d(R) = d$

3). $\exists$ an ideal of definition generated by d-elements but $\nexists$ any ideal of definition generated by $d-1$ elements.

Cor. R: local Noetherian ring, then $\dim R < \infty$.

Pf: We may calculate $\dim R$ by $\deg \chi_{I,R}$ ((2) above), which is $< \infty$. $\square$

Warning: $\exists$ Noetherian rings with $\dim = \infty$.

Note also that by Nakayama's lemma, $\dim_{k(m)} \frac{m}{m^2}$ = minimal # of generators of m as an ideal ($\geq \dim R$, by (3) above).

Def: If $\dim R$ = minimal # of generators of m, then R is called a regular local ring.

Example: $R = (k[x,y,z]/(xy, zw))_m$ $\quad m = (x, y, z, w)$

Claim: $R/(x+y, z+w)R$ is finite.

Indeed. $R/(x+y, z+w)R \cong (k[x,y,z,w]/(x+y, z+w, xy, zw))_m \cong (k[x,z]/(x^2, z^2))_{\overline{m}}$

$\cong k[x,z]/(x^2, z^2) \cong k \oplus kx \oplus kz \oplus kxz$

($\overline{m} = (x,z)$, anything like $a_0 + \Sigma a_{ij} x^i z^j$ $a_0 \neq 0$ is already a unit in $k[x,z]/(x^2, z^2)$).

Thus $\dim R \leq 2$. But $(xy, zw) \subseteq (x, z) \subseteq (x, z, y) \subseteq (x, y, z, w)$ exhibits it a length 2 chain, thus $\dim R = 2$.

Lemma: R: Noetherian ring.

(a). $x \in R$. $\mathfrak{p}, \mathfrak{q} \subseteq R$ primes s.t. $\mathfrak{p} \subseteq \mathfrak{p} + (x) \subseteq \mathfrak{q}$ and $\mathfrak{q}$ is a minimal such. Then there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$.

(b). If $x \in R$ and q minimal over x, then the height of q is either 0 or 1.
Pf: The primes between p and q are the primes of $R' = (R/p)_q = R_q/pR_q$. and qR' is the maximal ideal of R', which is also minimal over xR'. $\Rightarrow$ $\sqrt{xR'} = qR'$ and by previous lemmas, $\dim R' \leq 1$. "=" holds if $xR' \neq 0$, or $x \notin p$.
Part (b) follows from (a), by choosing any $q \in D(x)$. $\square$

Lemma: (R, m, k): Noetherian local ring. $x \in m$. Then $\dim R \leq \dim R/(x) + 1$. Equality holds if $x \notin$ any minimal prime of R.
Rmk: by prime avoidence, if $\dim R \geq 1$, such x that "=" holds always exists.
Pf: $R \twoheadrightarrow R/(x)$. Pick $\bar{x}_1, \cdots, \bar{x}_d \in R/(x)$ where $d = \dim R/(x)$ and $(\bar{x}_1, \cdots, \bar{x}_d)$ is an ideal of definition. $\Rightarrow$ $R/(x, x_1, \cdots, x_d) \cong R/(x) / (\bar{x}_1, \cdots, \bar{x}_d)$ and has finite length. $\Rightarrow$ $(x, x_1, \cdots, x_d)$ is an ideal of definition $\Rightarrow$ $\dim R \leq d+1$.
Moreover, prime chains that occur in $R/(x)$ are prime chains in R containing x. If $x \notin$ any minimal prime, such chains can be extended at least by one of the minimal primes. $\square$

Two lemmas on filtrations of modules.
Lemma 1: R: a ring, M: a finite R-module. Then $\exists$ a filtration of M by submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ s.t. $M_i/M_{i-1} \cong R/I_i$ for some I_i.
Pf: $M = (x_1, \cdots, x_r)$, then just set $M_i = (x_1, \cdots, x_i)$. $\square$

Lemma 2: R: Noetherian, M finite R-module. Then $\exists$ a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ s.t. $M_i/M_{i-1} \cong R/p_i$ for some prime p_i.
Pf: By lemma 1, it suffices to prove for $M = R/I$. Consider the set of ideals $\{J \supseteq I \mid \text{the conclusion fails for } R/J\}$. We will show that it's empty. Otherwise, it would contain a maximal element, say, J, s.t. R/J has no such filtration. Clearly J cannot be prime, otherwise $0 \subseteq R/J$ is a filtration. Then $\exists a, b \in R$ s.t. $ab \in J$ but $a, b \notin J$. Now $0 \longrightarrow \frac{aR+J}{J} \longrightarrow \frac{R}{J} \longrightarrow \frac{R}{aR+J} \longrightarrow 0$ is s.e. and we have $\frac{aR+J}{J} \cong \frac{R}{I'}$, $I' \supsetneq J$ since $b \in I'$ $\Rightarrow$ both I' and $aR+J$ strictly contains J thus R/I', $R/aR+J$ have a required filtration. $\Rightarrow$ R/J has a required

filtration. Contradiction. $\square$

Example: $R = M = k[x]$. we see that such filtrations are highly non-unique.
1. $0 \subseteq M$. 2. $0 \subseteq (x) \subseteq M$ 3. $0 \subseteq (x-a)(x) \subseteq (x) \subseteq M$ ...
However, as will be shown latter, the minimal primes occuring in any such a filtration will be unique with multiplicity!

Support of a module

Def: R: a ring. M an R-module. $\mathrm{Supp}(M) \triangleq \{\mathfrak{p} \in \mathrm{Spec}R \mid M_\mathfrak{p} \neq 0\}$.

Lemma: If M is finite, then $\mathrm{Supp}M$ is closed.

Pf: We show that, $\mathrm{Spec}R \setminus \mathrm{Supp}M$ is open. Indeed, let $M = (x_1, \cdots, x_r)$

$M_\mathfrak{p} = 0 \Rightarrow \frac{x_i}{1} = 0$, $i = 1, \cdots, r$ in $M_\mathfrak{p}$ $\Rightarrow \exists f_i \notin \mathfrak{p}$ s.t. $f_i x_i = 0$ in M

$\Rightarrow \exists f \notin \mathfrak{p}$ $f x_i = 0$ in M $(f = \prod_{i=1}^{r} f_i)$

$\Rightarrow M_f = 0$

$\Rightarrow M_\mathfrak{q} = 0$, $\forall \mathfrak{q} \in D(f)$

(A direct proof. M is an $R/\mathrm{ann}M$-module $\Rightarrow M_\mathfrak{p}$ is an $(R/\mathrm{ann}M)_\mathfrak{p}$-module.
$M_\mathfrak{p} \neq 0 \Rightarrow (R/\mathrm{ann}M)_\mathfrak{p} \neq 0 \Rightarrow \mathrm{ann}M \cdot R_\mathfrak{p} \neq R_\mathfrak{p} \Rightarrow \mathfrak{p} \supseteq \mathrm{ann}M$.
Conversely if $\mathfrak{p} \not\supseteq \mathrm{ann}M$, $\exists\, x \in \mathrm{ann}M$, $x \notin \mathfrak{p} \Rightarrow \frac{x}{1} \in R_\mathfrak{p}$ is a unit and $\frac{x}{1} M_\mathfrak{p} = 0$.
$\Rightarrow M_\mathfrak{p} = 0$) $\square$

Lemma: R: ring, M: R-module.
(1). If M is finite, then $\mathrm{Supp}(M/IM) = \mathrm{Supp}M \cap V(I)$
(2). $N \subseteq M \Rightarrow \mathrm{Supp}N \subseteq \mathrm{Supp}M$
(3). $M \twoheadrightarrow Q \Rightarrow \mathrm{Supp}Q \subseteq \mathrm{Supp}M$
(4). s.e.s: $0 \to N \to M \to Q \to 0 \Rightarrow \mathrm{Supp}M = \mathrm{Supp}N \cup \mathrm{Supp}Q$.

Pf: (2). (3). (4) follows directly from localization being exact.
For (1), since $M \twoheadrightarrow M/IM \Rightarrow \mathrm{Supp}(M/IM) \subseteq \mathrm{Supp}M$, and $(R/I)^{\oplus n} \twoheadrightarrow M/IM \Rightarrow$ $\mathrm{Supp}(M/IM) \subseteq V(I)$, where n is the number of a set of generators. Conversely, if $M_\mathfrak{p} \neq 0$ and $I \subseteq \mathfrak{p} \Rightarrow M_\mathfrak{p}/IM_\mathfrak{p} \neq 0$ by Nakayama's lemma $\Rightarrow (M/IM)_\mathfrak{p} \neq 0$. $\square$

Cor. Suppose $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M$, $M_i/M_{i-1}\cong R/\mathfrak{p}_i$, then all $\mathfrak{p}_i\in \operatorname{Supp} M$. $\square$

Lemma: $(R,\mathfrak{m},\kappa)$: local Noetherian ring. M: finite R-module.
Then $\operatorname{Supp} R=\{\mathfrak{m}\}\Leftrightarrow \ell_R(M)<\infty$.
Pf: "$\Rightarrow$": M has a finite filtration $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M$, $M_i/M_{i-1}\cong R/\mathfrak{m}$
$\Rightarrow \mathfrak{m}^n M=0\Rightarrow \ell_R(M)<\infty$.
"$\Leftarrow$": $\mathfrak{m}^n M=0$ for some $n\in\mathbb{N}$. If $\mathfrak{p}\subsetneq\mathfrak{m}$, then $\exists\, x\in\mathfrak{m}$. $x\notin\mathfrak{p}$. $x^nM=0$ but $x^n\in R_\mathfrak{p}$ is a unit $\Rightarrow M_\mathfrak{p}=0$. $\square$

Lemma:
M: an R-module with filtration $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_r=M$, $M_i/M_{i-1}\cong R/\mathfrak{p}_i$. Then the set of minimal elements in $\{\mathfrak{p}_i\}$ is the set of minimal elements in $\operatorname{Supp} M$.
Moreover we have $\#\{i\mid \mathfrak{p}_i=\mathfrak{p},\ \mathfrak{p}_i \text{ minimal}\}=\ell_{R_\mathfrak{p}}(M_\mathfrak{p})$.
Pf: For $M_\mathfrak{p}$, $0=M_{0\mathfrak{p}}\subseteq M_{1\mathfrak{p}}\subseteq\cdots M_{r\mathfrak{p}}=M_\mathfrak{p}$ is again a filtration, with $M_{i\mathfrak{p}}/M_{i-1\mathfrak{p}}\cong(R/\mathfrak{p}_i)_\mathfrak{p}\cong R_\mathfrak{p}/\mathfrak{p}_iR_\mathfrak{p}$, $\mathfrak{p}_iR_\mathfrak{p}$ is either $R_\mathfrak{p}$ or a prime ideal. Hence $M_\mathfrak{p}\neq 0\Leftrightarrow$ at least one of the inclusions is strict $\Leftrightarrow \mathfrak{p}\supseteq\mathfrak{p}_i$ for some $\mathfrak{p}_i$ minimal.
If $\mathfrak{p}$ is minimal, the filtration $0=M_{0\mathfrak{p}}\subseteq\cdots\subseteq M_{r\mathfrak{p}}=M_\mathfrak{p}$ is of successive quotients either $\kappa(\mathfrak{p}_i)$ or 0 $\Rightarrow \ell_{R_\mathfrak{p}}(M_\mathfrak{p})=\#\{\text{non-trivial inclusions}\}=\#\{i\mid M_i/M_{i-1}\cong R/\mathfrak{p}\}$. $\square$

Lemma:
R: Noetherian local ring, M a finite R-module, then $d(M)=\dim \operatorname{Supp}(M)$.
Pf: Take a filtration as above. We have $\dim\operatorname{Supp} M=\max\{\dim R/\mathfrak{p}_i\}$. But since $d(M)=\max\{d(R/\mathfrak{p}_i)\}$ and $d(R/\mathfrak{p}_i)=\dim R/\mathfrak{p}_i$, the result follows. $\square$

Associated primes.
Def. R: Noetherian and M finite. A prime $\mathfrak{p}$ is called an associated prime of the module M iff $\exists\, m\in M$, $\operatorname{Ann}(m)=\mathfrak{p}$ (or $R/\mathfrak{p}\hookrightarrow M$). $\operatorname{Ass}(M)=\{\mathfrak{p}\mid \mathfrak{p}$ associated primes of $M\}$.

Lemma:

R Noetherian and M finite. Take a filtration $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M$ s.t. $M_i/M_{i-1}\cong R/\mathfrak{p}_i$. Then $\mathrm{Ass}(M)\subseteq\{\mathfrak{p}_i\}$.

Pf: By induction on length of the filtration.

$n=1$. $M\cong R/\mathfrak{p}$. Any non-0 element in M has annihilator $\mathfrak{p}$.

Now suppose the lemma is true for modules with a filtration as above of length $\leq n-1$. Pick $m\in M$ whose annihilator is a prime $\mathfrak{p}$.

If $m\in M_{n-1}$, then we are done by induction hypothesis.

If $m\notin M_{n-1}$, then $0\neq\overline{m}\in M_n/M_{n-1}\cong R/\mathfrak{p}_n\Rightarrow\mathfrak{p}\subseteq\mathrm{Ann}\,\overline{m}=\mathfrak{p}_n$. If $\mathfrak{p}=\mathfrak{p}_n$ then we are done, otherwise $\mathfrak{p}\subsetneq\mathfrak{p}_n$. Pick $f\in\mathfrak{p}_n$, $f\notin\mathfrak{p}$. Consider the annihilator of $fm\in M_{n-1}$: $gfm=0\Rightarrow gf\in\mathfrak{p}$, $f\notin\mathfrak{p}\Rightarrow g\in\mathfrak{p}$; conversely any $g\in\mathfrak{p}$ kills fm.

$\Rightarrow\mathrm{Ann}(f\cdot m)=\mathfrak{p}$, we win by induction. $\square$

Cor. Condition as above, $\mathrm{Ass}(M)$ is finite. $\square$

Prop:

R Noetherian, M finite. Then the following sets of ideals coincide.

(1). Minimal primes in $\mathrm{Supp}(M)$

(2). Minimal primes in $\mathrm{Ass}(M)$

(3). Minimal primes in a filtration $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M$, $M_i/M_{i-1}\cong R/\mathfrak{p}_i$.

Pf: We have shown (1) = (3) already. Thus it suffices to show that, if $\mathfrak{p}$ is a (minimal) element in a filtration, then it is in (2), by the previous lemma.

Now, let $\mathfrak{p}\in\{\mathfrak{p}_i\}$ and i be minimal such that $\mathfrak{p}=\mathfrak{p}_i$. Pick $m\in M_i$, $m\notin M_{i-1}$, then $\mathrm{Ann}(m)\subseteq\mathfrak{p}_i$. Also $\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\mathfrak{p}_i m=0\Rightarrow\mathrm{Ann}(m)\supseteq\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\mathfrak{p}_i$. By our choice of i, $\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\not\subseteq\mathfrak{p}_i\Rightarrow\exists f\notin\mathfrak{p}_i$. $f\in\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}$ Consider $\mathrm{Ann}(fm)$. $\mathfrak{p}_i fm=0$ and $fm\notin M_{i-1}\Rightarrow\mathfrak{p}_i\subseteq\mathrm{Ann}(fm)$ and $\mathrm{Ann}(fm)\subseteq\mathfrak{p}_i\Rightarrow\mathrm{Ann}(fm)=\mathfrak{p}_i$. $\square$

Lemma:

R, M as above, then $\bigcup_{\mathfrak{p}\in\mathrm{Ass}(M)}\mathfrak{p}$ = set of zero divisors on M.

Pf: "$\subseteq$" is by definition

"$\supseteq$" Pick x a zero divisor. Let $0 \neq N = \{m \mid xm = 0\}$. $\Rightarrow$ $0 \neq \mathrm{Ass}(N) \subseteq \mathrm{Ass}(M)$

Take $n \in N$ s.t. $\mathrm{Ann}(n) = \mathfrak{q} \in \mathrm{Ann}M$, since $xn = 0$, $x \in \mathfrak{q}$. $\square$

Example: $R = M = k[x,y]/(x^2, xy)$

$\bar{x} \in M$ has annihilator $(\bar{x}, \bar{y})$, then one may guess that $\mathrm{Ass}(M) = \{(\bar{x}), (\bar{x}, \bar{y})\}$. Indeed we can find a filtration of M:

A line with a tangent

$$0 \subsetneq (\bar{x}) \subsetneq M$$

and $(\bar{x}) \cong \dfrac{R}{(\bar{x}, \bar{y})}$, $\dfrac{M}{(\bar{x})} \cong \dfrac{R}{(\bar{x})}$ the prop. above applies.

In particular, note that although $\mathrm{Spec}R \cong \mathrm{Spec}\,k[y]$ as topological spaces, this coarse invariant of $\mathrm{Ass}(R)$ shows that $\mathrm{Spec}R$ is not isomorphic to $\mathrm{Spec}\,k[y]$ as schemes.

§9. Regular Sequences and Depth.

Def: R: a ring. M an R-module. A sequence of elements $f_1, \cdots, f_r \in R$ is called M regular iff:

(1) f_i is a non-zero divisor on $M/(f_1, \cdots, f_r)M$

(2). $M/(f_1, \cdots, f_r)M \neq 0$.

If $I \subseteq R$ is an ideal and $f_1, \cdots, f_r \in I$ with (1) and (2), they are called an M-regular sequence in I. If $M = R$, it's just called a regular sequence in I.

The concept is not so well-behaved in sense of the following:

Example:

(1) $R = M = k[x, y, z]$ (a "global" ring)

$(x, y(1-x), z(1-x))$ is then regular; but $(y(1-x), z(1-x), x)$ is not, since in $R/(y(1-x))$, $z(1-x)$ is a zero-divisor, it kills $\bar{y} \neq 0$.

(2). $R = M = k[x, y, w_0, w_1, \cdots]/I$, $I = (yw_i, w_i - xw_{i+1}, \cdots)$ (Non-noetherian)

Claim: (x, y) is a regular sequence, but (y, x) isn't.

Indeed $R/(x) \cong k[y]$, but y a zero divisor in R.

Def: If (R, m) is a local ring, M an R-module. Then $\mathrm{depth}_m(M) \triangleq \sup$ of length of M-regular sequence in M.

Lemma:

Let R be Noetherian local, M finite over R. If $(x_1, \cdots, x_c)$ is a regular sequence, then so is any permutation: $(x_{i_1}, \cdots, x_{i_c})$.

Pf: Note that condition (2) forces $x_i \in m$.

It suffices to prove for 2 elements, since any permutation is a combination of transpositions. Now, take $x \in m$ a non-zero divisor, $y \in m$ a non-zero divisor on M/xM.

Consider the diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "x"] \arrow[d, "y"] & M \arrow[r] \arrow[d, "y"] & M/xM \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & M \arrow[r, "x"] \arrow[d] & M \arrow[r] \arrow[d] & M/xM \arrow[r] \arrow[d] & 0 \\
 & M/yM \arrow[r, "x"] \arrow[d] & M/yM \arrow[r] \arrow[d] & M/(x,y)M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\Rightarrow$ $0 \to \ker y \xrightarrow{x} \ker y \to 0 \to M/yM \xrightarrow{x} M/yM \to M/(x,y)M \to 0$ is exact

$\Rightarrow$ $m\ker y = \ker y$ $(x \in m)$ and $M/yM \xrightarrow{x} M/yM$ is an injection

$\Rightarrow$ $\ker y = 0$ by Nakayama and x is a non-zero divisor on M

$\Rightarrow$ (y, x) is regular. □

Lemma:

R: Noetherian local, M: finite R-module. $I \subseteq m$ an ideal of R. TFAE:

(1). $\exists\, x \in I$ which is not a zero-divisor on M, i.e. $\operatorname{depth}_I M \geq 1$

(2). We have $I \subsetneq q$ for all $q \in \operatorname{Ass}(M)$. (in particular $m \notin \operatorname{Ass} M$).

Pf: We know that $\bigcup_{q \in \operatorname{Ass} M} q = \{\text{zero-divisors on } M\}$

(2) $\Rightarrow$ (1) Since $\operatorname{Ass} M$ is a finite set, by prime avoidance, $\exists\, x \in I$, $x \notin q$, $\forall q \in \operatorname{Ass} M$

(1) $\Rightarrow$ (2) obvious □

Cor. In the previous lemma, if $I = m$, $\operatorname{depth}_m \geq 1 \Leftrightarrow m \in \operatorname{Ass} M$ □

Cor. If R Noetherian local, with residue field κ, then $\operatorname{depth}_R = 0$ iff there exists a non-zero module map $\kappa \hookrightarrow R$.

Pf: $\operatorname{depth} R = 0 \Leftrightarrow m \in \operatorname{Ass} M \Leftrightarrow \kappa \hookrightarrow R$. □

Example: $R = k[x,y]_{(x,y)}/(xy, x^2)$, then $x \neq 0$ and $mx = 0 \Rightarrow \operatorname{depth} R = 0$

Lemma:

R: Noetherian local, M: finite R-module. $\operatorname{depth}_m M \leq \dim(\operatorname{Supp} M) = d(M)$.

Pf: If M is of finite length, then $m \in \operatorname{Ass} M$, and $\operatorname{depth} M = 0$.

Now if M is not of finite length, and take any regular sequence on M, say $f_1, \cdots, f_r \in m$, we will show that its length $r \leq d(M)$ by induction on $d(M)$. The $d(M)=0$ case is done by the first sentence.

Since $(f_2, \cdots, f_r)$ is a regular sequence on M/f_1M, $r-1 \leq \operatorname{depth}(M/f_1M)$. It suffices to show that $d(M/f_1M) \leq \dim M - 1$, then $r-1 \leq \operatorname{depth}(M/f_1M) \leq d(M/f_1M) \leq d(M)-1$.

Since $0 \longrightarrow M \xrightarrow{f_1} M \longrightarrow M/f_1M \longrightarrow 0$ is exact, by the last lemma of §7, we know that $d(M/f_1M) = \deg \chi_{m, M/f_1M} < \deg \chi_{m,M} = d(M/f_1M)$. $\square$

Lemma:

R a ring and $J = (f_1, \cdots, f_c)$ generated by a regular sequence, then the graded ring $\bigoplus_{n \geq 0} J^n/J^{n+1}$ is graded isomorphic to $R/J[x_1, \cdots, x_c]$.

(A variant for modules also holds if $f_1, \cdots, f_c$ is a regular sequence on M, in which case the graded $\bigoplus_{n\geq 0} J^n/J^{n+1}$ – module $\bigoplus_{n \geq 0} J^nM/J^{n+1}M$ is graded isomorphic to $M/JM[x_1, \cdots, x_c]$).

$\operatorname{Spec} \bigoplus_{n \geq 0} J^n/J^{n+1} \longrightarrow \operatorname{Spec} R/J$ is called the normal cone of $\operatorname{Spec} R/J$ in $\operatorname{Spec} R$. c.f. the example below.

Example: $R = \mathbb{C}[x, y, z]/(x^3+y^3+z^3+x^{100}+z^{100}+y^{1000}) \supseteq J = (x, y, z)$

Then $J/J^2 \cong \mathbb{C}^3 = \mathbb{C}\{\bar{x}, \bar{y}, \bar{z}\}$. Surely $\Phi_n: \operatorname{Sym}^n(J/J^2) \twoheadrightarrow J^n/J^{n+1}$.

However, $\bar{x}^3 + \bar{y}^3 + \bar{z}^3 \in \operatorname{Ker} \Phi_3$ since $x^{100} + z^{100} + y^{1000} \in J^4$. $\Rightarrow R/J[\bar{x}, \bar{y}, \bar{z}] \ncong \bigoplus_{n \geq 0} J^n/J^{n+1}$, and (x, y, z) is not a regular sequence.

In fact $\bigoplus_{n \geq 0} J^n/J^{n+1} \cong \mathbb{C}[x, y, z]/(\bar{x}^3+\bar{y}^3+\bar{z}^3)$ and its max-spectrum is a cone in $\mathbb{C}^3$.

Proof of lemma.

We prove by induction on c. $c=1$ is clear.

Goal: $\sum_{|I|=n} a_I f^I \in J^{n+1} \Leftrightarrow a_I \in J$, $\forall I = (i_1, \cdots, i_c) \in \mathbb{Z}^n_{\geq 0}$.

We can write any element in J^{n+1} as $\sum_{|I|=n} b_I f^I$ with $b_I \in J$. Thus after substraction we are reduced to prove if $a=\sum_{|I|=n} a_I f^I = 0$ then $a_I \in J$, $\forall I$.

Collecting terms, we have $a=\sum_{e=0}^{n} (\sum_{|I'|=n-e} a_{I',e} f^{I'}) f_c^e$. $I'=(i_1,\cdots,i_c)$. Now we shall show by ascending induction on $0\le l\le n$ that if we have

$$\sum_{e=0}^{l} (\sum_{|I'|=n-e} a_{I',e} f^{I'}) f_c^e = 0$$

then $a_{I',e}\in J$. Set $J'=(f_1,\cdots,f_{c-1})$. $l=0$ is the induction hypothesis of $c-1$. Now $f_c(\sum_{e=1}^{l}(\sum_{|I'|=n-e} a_{I',e} f^{I'}) f_c^{e-1}) = -\sum_{|I'|=n} a_{I',e} f^{I'} \equiv 0 \mod J'$ and f_c is a non-zero divisor $\Rightarrow \sum_{e=1}^{l}(\sum$

$$f_c^{l}\Big(\sum_{|I|=n-l} a_{I',e} f^{I'}\Big) = -\sum_{e=0}^{l-1}\Big(\sum_{|I'|=n-l} a_{I',e} f^{I'}\Big) f_c^{e}$$

Collecting terms, we have $a=\sum_{e=0}^{n}(\sum_{|I'|=n-e} a_{I',e} f^{I'}) f_1^e$ where $I'=(0,i_1,\cdots,i_c)$. We will prove by induction that $\sum_{e=0}^{l}(\sum_{|I'|=l-e} a_{I',e} f^{I'}) f_1^e \in J^{l+1}$, then $a_{I',e}\in J$

Now $\sum_{|I'|=l} \overline{a_{I',0}}\, \bar{f}^{I'} \in \bar{J}^{l+1}$ mod(f_1), and since $(f_2,\cdots,f_c)$ is a regular sequence mod(f_1), by induction on the length of sequence, $\overline{a_{I',0}}\in \bar{J} \Rightarrow a_{I',0}\in J$. It follows that $\sum_{e=1}^{l}(\sum_{|I'|=l-e} a_{I',e} f^{I'}) f_1^e \in J^{l+1} \Rightarrow f_1(\underbrace{\sum_{e=0}^{l-1}(\sum_{|I'|=l-e-1} a_{I',e} f^{I'}) f_1^e}_{\text{mod}}) \in J^{l+1}$

$$0=f_1a_1+a_2f_2+\cdots+a_cf_c$$

$$= \sum_{e=0}^{l}\Big(\sum_{|I'|=l-e} b_{I',e} f^{I'}\Big) f_1^e$$

$$\sum_{|I'|=l} \overbrace{b}^{J}_{I',0}\, f^{I'} \in \bar{J}^{l+1}$$

mod(f_1) $\quad \sum_{|I'|=l}(\overline{a_{I',e}} f^{I'}) = 0 \Rightarrow \overline{a_{I',e}} \in \bar{J}$

$$\Rightarrow a_{I',e} = b_{I',1} f_1+\cdots+b_{I',c} f_c \quad |I'|=l$$

$$\Big(\sum_{e=0}^{l-1}\Big(\sum_{|I'|=l-e} a_{I',e} f^{I'}\Big) f_1^{e-1}\Big) = 0$$

- Digression: Homological Algebra

Lemma: R a ring, M an R-module.

(1). There exists an exact complex $\cdots F_2 \to F_1 \to F_0 \to M \to 0$, where F_i are free R-modules.

(2). If R is Noetherian and M finite, then we can choose F_i finite free. $\square$

Notation: We will denote the complex $\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots$ by $F_\bullet$.

1). The i-th homology group is defined as $H_i(F_\bullet) = \ker d_i / \operatorname{Im} d_{i+1}$.

2). Morphisms of complexes $\alpha: F_\bullet \to G_\bullet$ is given by maps $\alpha_i: F_i \to G_i$ such that

$$\begin{tikzcd} \cdots \arrow[r] & F_i \arrow[r, "d_i"] \arrow[d, "\alpha_i"] \arrow[dr, phantom, "\circlearrowright"] & F_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] & \cdots \\ \cdots \arrow[r] & G_i \arrow[r, "d_i"] & G_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

and α induces maps on homology groups: $H_i(\alpha): H_i(F_\bullet) \to H_i(G_\bullet)$

3). $\alpha, \beta: F_\bullet \to G_\bullet$ morphisms of complexes, then a homotopy between α and β is a collection of maps $F_i \to G_{i+1}$ s.t. $\alpha_i - \beta_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$

$$\begin{tikzcd} \cdots \arrow[r] & F_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & F_i \arrow[r, "d_i"] \arrow[d] \arrow[dl, "h_i"'] & F_{i-1} \arrow[r] \arrow[d] \arrow[dl, "h_{i-1}"'] & \cdots \\ \cdots \arrow[r] & G_{i+1} \arrow[r, "d_{i+1}"] & G_i \arrow[r, "d_i"] & G_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

Immediate consequence: $\alpha \simeq \beta \Rightarrow H_\bullet(\alpha) = H_\bullet(\beta)$ on homology groups.

4). Cohomological notation: $F^\bullet: \cdots \to F^i \xrightarrow{d^{i+1}} F^{i+1} \to \cdots$, $H^i(F^\bullet) = \ker d^{i+1} / \operatorname{im} d^i$

Lemma: Suppose $G_\bullet \to N \to 0$ is an exact resolution, $F_\bullet \to M \to 0$ is a free resolution, and $\gamma: M \to N$ is an R-map, then:

(1). $\exists\, \alpha: F_\bullet \to G_\bullet$ which induces γ, i.e. $H_0(\alpha) = \gamma$.

(2). Any two such maps are homotopic. $\square$

Def: ($\operatorname{Ext}^i_R(M,N)$) Choose a free resolution $F_\bullet$ of M. Consider the complex:

$\operatorname{Hom}_R(F_\bullet, N): 0 \to \operatorname{Hom}_R(F_0, N) \xrightarrow{d^1} \operatorname{Hom}_R(F_1, N) \xrightarrow{d^2} \operatorname{Hom}_R(F_2, N) \to \cdots$

Set: $\operatorname{Ext}^i(M,N) = H^i(\operatorname{Hom}(F_\bullet, N))$. It's "the" concept by the next lemma:

Lemma: R a ring, M_1, M_2, N are R-modules. $F_\bullet$ a free resolution of M_1, $G_\bullet$ a free resolution of M_2. $\varphi: M_1 \to M_2$, and $\alpha: F_\bullet \to G_\bullet$ any map of complexes s.t. $H_0(\alpha) = \varphi$. Then the induced maps:

$$H^i(\alpha): H^i(\mathrm{Hom}_R(F_\bullet, N)) \longrightarrow H^i(\mathrm{Hom}_R(G_\bullet, N))$$

are independent of choices of α. Furthermore, if φ is an isomorphism, so is each $H^i(\alpha)$; if $M_1 = M_2$, $\varphi = id$, so is $H^i(\alpha) = id$. $\square$

Rmk: This lemma says that $\mathrm{Ext}^i(M,N)$ form a contravariant functor in the variable M.

Lemma: $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$. $\square$

Lemma: $M \to M'$ gives $\mathrm{Ext}^i(M',N) \to \mathrm{Ext}^i(M,N)$; $N \to N'$ gives $\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,N')$. Thus $\mathrm{Ext}^i_R : (\mathrm{Mod}_R)^{opp} \times (\mathrm{Mod}_R) \longrightarrow \mathrm{Mod}_R$ $\square$

Lemma: R a ring, M an R-module. $0 \to N' \to N \to N'' \to 0$ s.e.s. of R-modules $\Rightarrow \exists$ long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(M,N') \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N'') \\ &\longrightarrow \mathrm{Ext}^1_R(M,N') \longrightarrow \cdots \\ &\longrightarrow \mathrm{Ext}^i_R(M,N') \longrightarrow \mathrm{Ext}^i_R(M,N) \longrightarrow \mathrm{Ext}^i_R(M,N'') \\ &\longrightarrow \mathrm{Ext}^{i+1}_R(M,N') \longrightarrow \cdots \end{aligned}$$

Example:

a). $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = ?$

$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \to 0$ is a free resolution

$\Rightarrow$ The homology of: $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{p} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \to 0$ computes $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z})$, which is the same sequence as $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow 0$

$\Rightarrow \mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) (= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z})) = 0$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) = \mathbb{Z}/p$, and the higher groups $\mathrm{Ext}^i(\mathbb{Z}/p, \mathbb{Z}) = 0$, $i > 1$.

b). $R = k[\varepsilon] \triangleq k[x]/(x^2)$. i.e. $\varepsilon^2 = 0$. $\mathrm{Ext}^i_R(k,k) = ?$

$F_\cdot : \cdots \longrightarrow R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \longrightarrow k \longrightarrow 0$

$\Rightarrow \mathrm{Hom}_R(F_\cdot, k) : 0 \longrightarrow k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} \cdots$

$\Rightarrow \mathrm{Ext}^i_R(k,k) = k$ for all $i > 0$.

Lemma: R a ring, M, N R-modules, $x \in R$ s.t. either $xM = 0$ or $xN = 0$. Then x annihilates each $\mathrm{Ext}^i_R(M,N)$. $\square$

This follows immediately once one realizes that for all $x \in R$, multiplication on $\mathrm{Ext}^i_R(M,N)$ is the same as the map induced by $M \xrightarrow{\cdot x} M$ or $N \xrightarrow{\cdot x} N$ by functoriality).

Using this language, we have a neat interpretation of depth:

Lemma: R Noetherian, local, with maximal ideal m. M a finite R-module. $M \neq 0$. Then $\mathrm{depth}_R(M) = \min\{i \mid \mathrm{Ext}^i_R(R/m, M) \neq 0\}$.

Pf: By a previous lemma, $\mathrm{depth}_R(M) = 0 \Leftrightarrow 0 \neq \mathrm{Hom}(R/m, M) = \mathrm{Ext}^0(R/m, M)$.
Thus we may assume that $\mathrm{depth}(M) > 0$, and prove by induction. Now, $\exists x \in m$, x not a zero divisor on M s.t. $\mathrm{depth}(M/xM) = \mathrm{depth}(M) - 1$. (Just take $x = f_1$, where $f_1, \cdots, f_d$ is a maximal regular sequence). Consider the s.e.s.

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

$\Rightarrow$ l.e.s.

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(K, M) \xrightarrow{0} \mathrm{Hom}(K, M) \longrightarrow \mathrm{Hom}(K, M/xM) \\ &\longrightarrow \mathrm{Ext}^1(K, M) \xrightarrow{0} \mathrm{Ext}^1(K, M) \longrightarrow \mathrm{Ext}^1(K, M/xM) \\ &\longrightarrow \cdots \qquad\qquad \cdots \longrightarrow \mathrm{Ext}^{d-2}(K, M/xM) \\ &\longrightarrow \mathrm{Ext}^{d-1}(K, M) \xrightarrow{0} \mathrm{Ext}^{d-1}(K, M) \longrightarrow \mathrm{Ext}^{d-1}(K, M/xM) \\ &\longrightarrow \mathrm{Ext}^{d}(K, M) \xrightarrow{0} \cdots \end{aligned}$$

By induction hypothesis, $\mathrm{Ext}^i(K, M/xM) = 0$ $i < d-1$ and $\mathrm{Ext}^{d-1}(K, M/xM) \neq 0$

l.e.s $\Rightarrow$ $\mathrm{Ext}^i(K, M) = 0$, $i \leq d-1$ and $0 \neq \mathrm{Ext}^{d-1}(K, M/xM) \cong \mathrm{Ext}^d(K, M)$ $\square$

This kind of "dimension shifting" is constantly see in AG and CA.

Cor. R: Noetherian local ring. $0 \to N' \to N \to N'' \to 0$ a s.e.s. of finite R-modules. Then:

(1). $\operatorname{depth}_R N'' \geq \min\{\operatorname{depth}_R N, \operatorname{depth}_R N' - 1\}$

(2). $\operatorname{depth}_R N' \geq \min\{\operatorname{depth}_R N, \operatorname{depth}_R N'' + 1\}$.

Pf: The s.e.s gives rise to a l.e.s:

$$
\begin{aligned}
&\cdots \longrightarrow \operatorname{Ext}^{i-1}(K, N'') \\
&\to \operatorname{Ext}^i(K, N') \to \operatorname{Ext}^i(K, N) \to \operatorname{Ext}^i(K, N'') \\
&\to \operatorname{Ext}^{i+1}(K, N') \to \operatorname{Ext}^{i+1}(K, N) \to \operatorname{Ext}^{i+1}(K, N'') \\
&\to \cdots
\end{aligned}
$$

Thus, we can see that as long as $\operatorname{Ext}^i(K,N) = 0 = \operatorname{Ext}^{i+1}(K, N')$, $\operatorname{Ext}^i(K, N'') = 0$; and as long as $\operatorname{Ext}^i(K,N) = 0 = \operatorname{Ext}^{i-1}(K, N'')$, $\operatorname{Ext}^i(K, N') = 0$. The result follows from the previous lemma $\square$

In the next section, we will see that higher depth means "better". This cor. shows that the modules in a resolution are "better" going to the left.

Rmk: We defined depth by taking $\operatorname{Ext}^i_R(M,N) = H^i(\operatorname{Hom}(F_\bullet, N))$ for a projective (free) resolution of M: $F_\bullet \to M \to 0$. However, it can be defined by taking an injective resolution of N: $0 \to N \to I^\bullet$, and $\operatorname{Ext}^i_R(M,N) = H^i(\operatorname{Hom}(M, I^\bullet))$.

One can prove that these definition agree by showing that both are equal to $H^i(\operatorname{Tor}(\operatorname{Hom}(F_\bullet, I^\bullet)))$.

§10. Cohn-Macaulay Modules

Def. R: Noetherian local ring. M a finite R-module. We say that M is Cohn-Macaulay if $\dim \operatorname{Supp} M = \operatorname{depth}_R M$.

Notation: In situation of the def., let $f_1, \cdots, f_d$ be an M-regular sequence with $d = \dim(\operatorname{Supp} M) = \operatorname{depth}_R(M)$. Then we say that $g \in \mathfrak{m}$ is good w.r.t. $(M; f_1, \cdots, f_d)$ if for all $i = 0, 1, \cdots, d-1$, we have $\dim(\operatorname{Supp} M \cap V(g, f_1, \cdots, f_i))(= \dim \operatorname{Supp}(M/(g, f_1, \cdots, f_i)M)) = d - i - 1$.

Lemma: In the situation as above

(1). g is a non-zero divisor

(2). M/gM is CM with a maximal regular sequence $(f_1, \cdots, f_{d-1})$.

Pf: By induction on d $(d > 0)$.

$d = 1$. Only need to show that g is a non-zero divisor, since by assumption of g being good, $\dim \operatorname{Supp} M \cap V(g) = 0$, and dim 0 modules are trivially CM.

Let $K = \ker(M \xrightarrow{g} M)$. $\operatorname{Supp} K \subseteq \operatorname{Supp} M \cap V(g)$, and $\dim K = 0$. If $K \neq 0$, then we have $\{\mathfrak{m}\} = \operatorname{Ass} K \subseteq \operatorname{Ass} M \Rightarrow \operatorname{depth} M = 0$, contradiction.

$d > 1$. Consider the diagram: (still let $\ker(M \xrightarrow{g} M) = K$, $\ker(M/f_1M \xrightarrow{g} M/f_1M) = K_1$)

$$\begin{array}{ccccccccc}
0 \to & M & \xrightarrow{f_1} & M & \longrightarrow & M/f_1M & \to 0 \\
 & \downarrow g & & \downarrow g & & \downarrow g & \\
0 \to & M & \xrightarrow{f_1} & M & \longrightarrow & M/f_1M & \to 0
\end{array}$$

Snake lemma $\Rightarrow$ $0 \to K \xrightarrow{f_1} K \to K_1 \to M/gM \xrightarrow{f_1} M/gM \to M/(f_1, g)M \to 0$. By assumption g is good w.r.t. M/f_1M (which is still CM), and the regular sequence $f_2, \cdots, f_d$. Ind. hypo $\Rightarrow K_1 = 0$ and $K \underset{\cong}{\xrightarrow{f_1}} K$. Nakayama $\Rightarrow K = 0$ and g is a non-zero divisor. Moreover $0 \to M/gM \xrightarrow{f_1} M/gM$ says that f_1 is a non-zero divisor on M/gM. Again by ind. hypo. $M/(f_1, g)M$ is CM with regular sequence $f_2, \cdots, f_{d-1}$. Hence M/gM is CM with a regular sequence $f_1, \cdots, f_{d-1}$. $\square$

From the proof, we see that CM allows us to cut dimension down and do induction. This argument will appear often.

Lemma: R Noetherian local. M: CM over R. If $g \in \mathfrak{m}$ s.t. $\dim \operatorname{Supp} M/gM = \dim \operatorname{Supp} M - 1$. Then:

1). g is a non-zero divisor

2). M/gM is CM of depth 1 less.

Pf: Pick a maximal regular sequence $f_1, \cdots, f_d$ ($d \geq 1$ by assumption, and the lemma is always true if $d=1$, by the proof of the previous lemma. g is a non-zero divisor, and since $\dim \operatorname{Supp} M/gM = 0 \Rightarrow$ CM. Thus we may start with $d \geq 2$).

If g is good w.r.t $(M; f_1, \cdots, f_d)$ then we are done by the previous lemma. Now pick $h \in \mathfrak{m}$ s.t. 1). h is good w.r.t. $(M, f_1, \cdots, f_d)$

2). $\dim \operatorname{Supp} M \cap V(h, g) = d-2$

This is possible because if $\{q_i\}$ is the set of minimal primes of $\operatorname{Supp} M$, $\operatorname{Supp} M/gM$ and $\operatorname{Supp} M/(f_1, \cdots, f_i)M$, $i = 1, \cdots, d-1$, then $q_i \neq \mathfrak{m}$ since none of these modules are Artinian, and prime avoidance applies to provide an h.

It follows from the previous lemma that $(h, f_1, \cdots, f_{d-1})$ is a new regular sequence and g is good w.r.t. (M, h). (i.e. g cuts dimension down on M/hM). Now we repeat this process, to obtain an h' s.t. 1) h' is good w.r.t. $(M; h, f_1, \cdots, f_{d-1})$

2). $\dim \operatorname{Supp} M \cap V(h', h, g) = d-3$

$\cdots$ $(d-1)$ times later $\cdots$

We replace $(f_1, \cdots, f_d)$ by another regular sequence w.r.t. which g is good. Now the previous lemma applies.

A non-constructive proof. $\cdots$ (after obtaining h) $\cdots$

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "h"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "g"] & M/hM \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r, "h"] & M \arrow[r] & M/hM \arrow[r] & 0
\end{tikzcd}
$$

Now we finish the proof by induction on d. Since M/hM is CM of dim $d-1$

and $\dim M/(g,h)M = d-2$. Ind. hypo. $\Rightarrow$ g is a non-zero divisor on M/hM and $M/(g,h)M$ is CM of dim $d-2$.

Snake lemma $\Rightarrow$ $0 \to K \xrightarrow{g} K \to 0$ and $0 \to M/gM \xrightarrow{h} M/gM \to M/(g,h)M \to 0$
$\Rightarrow K=0$ by Nakayama and M/gM is CM of dim $d-1$ as h is a non-zero divisor on M/gM. $\square$

de Jong likes the constructional way better, "the right way of doing things"...

By this lemma, if we can take a sequence of elements that really cuts down dimensions, then:

Cor. R: Noetherian local. M: CM over R. If $\exists\, g_1,\cdots,g_c$ s.t. $\dim M/(g_1,\cdots,g_c)M = \dim M - c$, then $g_1,\cdots,g_c$ is a regular sequence and can be extended to a maximal regular sequence. $\square$

Def. A Noetherian local ring is called CM if it's CM as a module over itself.

Lemma: If R is local Noetherian CM of dim d, then any maximal chain of primes $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n$ has length $n=d$.

Pf: If $\dim R = 0$, the result is true trivially. Now we prove by induction

Here $\mathfrak{p}_0 \in \{\mathfrak{q} \mid \mathfrak{q}$ a minimal prime of $R\} = \{\mathfrak{q}_1,\cdots,\mathfrak{q}_r\}$. By assumption $\mathfrak{p}_1 \notin \{\mathfrak{q}\}$ and prime avoidance $\Rightarrow \exists\, x \in \mathfrak{p}_1,\ x \notin \mathfrak{q}_i$

Hence $\dim R/xR = d-1$ and thus is CM by the previous lemma. Ind. hypo $\Rightarrow$ the maximal chain $\mathfrak{p}_1/(x) \subseteq \cdots \subseteq \mathfrak{p}_n/(x)$ has length $n-1 = d-1$. $\Rightarrow n=d$. $\square$

Application: $k[x_1,\cdots,x_d]$ has dimension d. ($k=\bar{k}$)

Indeed $k[x_1,\cdots,x_d]_{(x_1,\cdots,x_d)}$ is CM of dim d ($x_1,\cdots,x_d$ is a regular sequence).
$\Rightarrow$ Every maximal chain of prime ideals in $k[x_1,\cdots,x_d]_{(x_1,\cdots,x_d)}$ has length d.
Now any maximal chain of prime ideals ends with $(x_1-a_1,\cdots,x_d-a_d)$, which is isomorphic to $(x_1,\cdots,x_d)$ by translation.

$\Rightarrow$ Every maximal chain of prime ideals in $k[x_1,\cdots,x_d]$ has length d.
or $\operatorname{Spec} k[x_1,\cdots,x_d]$ is equidimensional of dim d!

Lemma:

R local Noetherian CM ring, $\mathfrak{p} \subseteq R$ prime, $\dim R = \dim R_\mathfrak{p} + \dim R/\mathfrak{p}$.

Pf: Pick $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_r \subseteq R_\mathfrak{p}$, $\bar{\mathfrak{p}}_0 \subseteq \cdots \subseteq \bar{\mathfrak{p}}_s \subseteq R/\mathfrak{p}$ maximal chains of primes.
Then automatically, $\mathfrak{q}_r = \mathfrak{p}R_\mathfrak{p}$, $\bar{\mathfrak{p}}_0 = (0)$

$\Rightarrow$ $\mathfrak{q}_0 \cap R \subseteq \cdots \subseteq \mathfrak{q}_r \cap R = \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_s$ is a maximal chain of primes in R. By the previous lemma, $\dim R_\mathfrak{p} + \dim R/\mathfrak{p} = r+s = n = \dim R$. $\square$

Caution: This result even may not hold for rings that are localizations of finite type rings over $\mathbb{C}$! Reason: $\operatorname{Spec} R$ needs not be equidimensional.

Lemma:

R: local Noetherian CM ring. $\mathfrak{p} \subseteq R$ prime $\Rightarrow R_\mathfrak{p}$ CM.

Pf: Let $c = \dim R_\mathfrak{p}$. We want to construct a regular sequence of length c in $R_\mathfrak{p}$ (Note that it suffices to check for $c>0$; $c=0 \Rightarrow$ Artinian, which is automatically CM).

For this purpose, it's good enough to find $f_1,\cdots,f_c \in \mathfrak{p}$, an R-regular sequence in $\mathfrak{p}$. Indeed, since localizations are exact:

$$0 \to R \xrightarrow{f_1} R \;\Rightarrow\; 0 \to R_\mathfrak{p} \xrightarrow{f_1} R_\mathfrak{p} \;\Rightarrow\; f_1 \text{ non-zero divisor}$$

$$\cdots$$

$$0 \to R/(f_1,\cdots,f_{i-1}) \xrightarrow{f_i} R/(f_1,\cdots,f_{i-1}) \Rightarrow 0 \to R_\mathfrak{p}/(f_1,\cdots,f_{i-1}) \xrightarrow{f_i} R_\mathfrak{p}/(f_1,\cdots,f_{i-1})$$

Now R CM, for f_1 in $\mathfrak{p}$ to be a non-zero divisor, it's necessary and sufficient that $f_1 \notin$ any minimal prime of R. This is possible since $c>0 \Rightarrow$ $\mathfrak{p}$ is not any prime ideal $\Rightarrow \exists f_1 \in \mathfrak{p}$, $f_1 \notin \mathfrak{q}_i$, $\forall \mathfrak{q}_i$ minimal prime in R.

Next R/f_1R is again a CM-module, if $\dim(R/f_1R) = 0$ then we are done. Otherwise, it's necessary and sufficient that $f_2 \notin$ any minimal prime in $V(f_1)$ (By the previous lemma, all such minimal primes have $\dim R_\mathfrak{q} = 1$ thus $\mathfrak{p} \nsubseteq \mathfrak{q}$ for all these $\mathfrak{q}$'s). Such an f_2 exists by prime avoidance.

Repeat this argument c times. $\square$

Def. A Noetherian ring R is called Cohn-Macaulay if all of its local rings are CM.

Rmks: 1). Any localization of a CM ring is CM.

2). In general, a finitely generated algebra over a CM ring need not be CM. For example k: a field $\Rightarrow$ CM, but $k[x,y]/(x^2, xy)$ is not CM.

Lemma: R CM $\Rightarrow R[x]$ CM.

Rmk: In general, $\dim R[x] \neq \dim R + 1$ for Noetherian rings.

Pf: Pick $q \subseteq R[x]$ a prime over $p \subseteq R$, i.e. $p = q \cap R$. Pick a regular sequence $f_1, \cdots, f_d \in R_p$.

Claim: $f_1, \cdots, f_d$ are regular in $R[x]_q$.

Indeed, $R_p[x]$ is free over R_p as an R_p-module thus similar as in the previous lemma, $f_1, \cdots, f_d$ are regular in $R_p[x]$

$\Rightarrow f_1, \cdots, f_d$ are $R[x]_q$ regular since $R[x]_q$ is a further localization of $R_p[x]$.

(This also follows from a general fact that $R_p \longrightarrow R[x]_q$ is flat, and under flat morphisms, regular sequences $\longmapsto$ regular sequences)

Now, consider $R[x]_q/(f_1, \cdots, f_d)R[x]_q \cong (R/(f_1, \cdots, f_d)[x])_{\bar{q}}$, where $\bar{q}$ is the image of q in $R/(f_1, \cdots, f_d)[x]$. It suffices to show that, for Artinian local R' $(R/(f_1, \cdots, f_d))$, $R'[x]_q$ is CM for any $q \subseteq R'[x]$. This is the content of the next lemma. In both cases of the lemma, $f_1, \cdots, f_d$ can be completed into a regular sequence and $R/(f_1, \cdots, f_d, f)[x]_q$ or $R/(f_1, \cdots, f_d)[x]_q$ are Artinian, thus of dim 0.

$\square$

Lemma:

R: Artinian local ring, then $\forall q$ prime in $R[x]$, $R[x]_q$ is CM.

Pf: There are two cases to consider. (i) q contains a monic polynomial

(ii) q doesn't contain any monic polynomial.

In case (i) f is a non-zero divisor and $R[x]/(f)$ is finite free over R, so that $R[x]/(f)$ is Artinian, thus CM of dim 0. Hence $R[x]_q$ is CM of dim 1.

(ii). In this case $q = mR[x]$. Indeed $q \cap R = m \Rightarrow \kappa(m)[x] \twoheadrightarrow R[x]/q$, which is a quotient of $\kappa(m)[x]$. If $R[x]/q \neq \kappa(m)[x]$ then $R[x]/q \cong \kappa(m)[x]/(\bar{f})$ an irreducible polynomial in $\kappa(m)[x]$. $f = a_M x^M + \cdots + a_{N+1} x^{N+1} + x^N + \cdots$ $a_i \in m$, $i \geq N+1$. But since $m \subseteq q$, we may well remove all the $a_M, \cdots, a_{N+1}$ since $a_k x^k \in q$, $M \geq k \geq N+1$ and we end up with a monic polynomial. Contradiction. Thus $\bar{f} = 0$ and $q = mR[x]$. Moreover, q consists of nilpotent elements $\Rightarrow \operatorname{Spec} R[x] \cong \operatorname{Spec} R[x]/q = \operatorname{Spec} \kappa(m)[x]$. and is irreducible. Hence q is minimal and $\dim R[x]_q = 0$, thus CM.

$\square$

Lemma: k (a field) is CM, $\mathbb{Z}$ is CM.

Pf: k is Artinian; $\mathbb{Z}_p$ is dim 1, and has p regular. $\square$

Lemma: $k[x_1, \cdots, x_n]$ and $\mathbb{Z}[x_1, \cdots, x_n]$ are CM. $\square$

Lemma: (Dimension Shifting)

R: Noetherian local ring, CM of dim d. M: finite R module. Then for any s.e.s. $0 \to K \to R^{\oplus n} \to M \to 0$, we have either $\operatorname{depth} K \geq \operatorname{depth} M$ or $\operatorname{depth} M = d$.

Pf: By a previous lemma, $\operatorname{depth} K \geq \min\{\operatorname{depth} R, \operatorname{depth} M + 1\}$. $\square$

Def. R: Noetherian local CM ring. A finite module M/R is called maximal CM iff $\operatorname{depth} M = \operatorname{depth} R$.

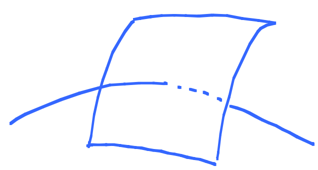
Lemma:

R: Noetherian local, CM of dim d. M any finite module. Then $\exists$ an exact complex $0 \to K \to F_i \to F_{i-1} \to \cdots \to F_0 \to M \to 0$ with F_j free and K MCM. More precisely, if $\operatorname{depth} M = c$, then $i = d - c - 1$. $\square$

This lemma is useful if we want to know wether $K_0(R) \to K_0(R') \to 0$. We may start by resolving M's with projective (free) resolutions, and thus only need to worry about MCM modules and check under what conditions they are free. (regular R, Serre)

Catenary Rings

Def: A ring is said to be catenary if $\forall$ primes $\mathfrak{p} \subseteq \mathfrak{q}$, all maximal chains of primes $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_e = \mathfrak{q}$ have the same finite length.



Not catenary, but we would like all irreducible components to be catenary.

Lemma:

Any localization of a catenary ring is catenary. The same holds for quotient. $\square$

Caution: $\exists$ example of a Noetherian catenary ring R and a finite type R-algebra which is not catenary, i.e. $\exists$ $R[x_1, \cdots, x_c]$ not catenary, by the previous lemma.

Def: R is called universally catenary if $R[x_1, \cdots, x_c]$ are catenary, $\forall c \geq 0$. Then it follows from the previous lemma that any R-algebra is catenary and thus converse is trivially true.

Lemma:

A Noetherian CM ring is u.c. Thus $\mathbb{Z}$, k are u.c.

Pf: $R[x_1, \cdots, x_n]$ is CM if R is. The result follows by localizing at $\mathfrak{q}$ and the lemma that any CM local ring has maximal chains of primes equal length $\square$

§11. Regular Rings

Def: (R, m, κ) a Noetherian local ring is called regular iff $\dim_\kappa m/m^2 = \dim R$.

Lemma:

If (R, m, κ) is regular, then the graded ring $\oplus_{n\geq 0} m^n/m^{n+1}$ is isomorphic to the graded polynomial ring $\kappa[X_1, \cdots, X_d]$, where $d = \dim R$. The converse is trivially true.

Pf: Let $x_1, \cdots, x_d$ be a minimal set of generators of the ideal m. Then there exists a surjection $\kappa[X_1, \cdots, X_d] \longrightarrow \oplus_{n\geq 0} m^n/m^{n+1}$, $X_i \mapsto x_i \in m/m^2$.

Thus it suffices to check injectivity. $\dim R = d \Rightarrow (n \mapsto \dim_\kappa m^n/m^{n+1})$ is a numerical polynomial of deg $d-1$. Meanwhile, the numerical polynomial for the polynomial ring $\kappa[X_1, \cdots, X_d]$ is $\varphi(m) = \binom{m+d-1}{d-1}$. If $0 \neq g$, homogeneous of degree c, is in the kernel, then $\kappa[X_1, \cdots, X_d]/(g) \twoheadrightarrow \oplus_{n\geq 0} m^n/m^{n+1}$, but the numerical polynomial for $\kappa[X_1, \cdots, X_d]/(g)$ is $\binom{m+d-1}{d-1} - \binom{m-c+d-1}{d-1}$, which has degree $d-2$, contradiction. It follows that the map must be an isomorphism. $\square$

Lemma:

Any regular local ring is a domain.

Pf: If not, take $0 \neq a, b \in R$, $a \cdot b = 0$. Consider the filtration $R \supseteq m \supseteq m^2 \supseteq \cdots$

Take the smallest i so that $a \in m^i$, $a \notin m^{i+1}$; k so that $b \in m^k$, $b \notin m^{k+1}$.

(such i, k exist by Artin-Rees lemma).

$\Rightarrow 0 \neq \bar{a} \in m^i/m^{i+1}$, $0 \neq \bar{b} \in m^k/m^{k+1}$. But $ab = 0 \Rightarrow ab \equiv 0 \mod m^{k+i+1}$

$\Rightarrow \bar{a} \cdot \bar{b} = 0$, contradiction with $\oplus_{n\geq 0} m^n/m^{n+1}$ being a domain. $\square$

In fact, this lemma easily generalizes to the case where R Noetherian and $\oplus_{n\geq 0} m^n/m^{n+1}$ is a domain (e.g. R CM)

Lemma.

R: regular local ring, $\{x_1, \cdots, x_d\}$ a minimal set of generators of m, (a regular system of parameters). Then $\{x_1, \cdots, x_d\}$ is a regular sequence, and $R/(x_1, \cdots x_c)$

is a regular local ring of dim $d-c$. In particular, R is CM.

Pf: x_1 is a non-zero divisor since R is regular. Moreover, R/x_1R has $(\bar{x}_2, \cdots, \bar{x}_d)$ generating $\bar{m}$. But $\dim R/x_1R = d-1$, thus R/x_1R is regular local by definition. An induction finishes the proof. ($d=0$ case is trivial). $\square$

Question: R regular local $\Rightarrow R_p$ regular local?

Lemma.

R: Noetherian local ring, $x \in m$. M a finite module such that x is a non-zero divisor on M and M/xM is free over R/xR. Then M is free over R. (x is necessarily a non-zero divisor in R)

Pf: $0 \to M \xrightarrow{x} M \to M/xM \to 0$ where $M/xM \cong (R/xR)^{\oplus r}$. Pick $m_1, \cdots, m_r$ which are mapped basis elements of M/xM. By Nakayama, $m_1, \cdots, m_r$ generate M. Thus we have a diagram: ($K = \ker\varphi$)

$$\begin{tikzcd} & R^{\oplus r} \arrow[r, "x"] \arrow[d, "\varphi"] & R^{\oplus r} \arrow[r] \arrow[d, "\varphi"] & (R/xR)^{\oplus r} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r, "x"] & M \arrow[r] & M/xM \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow K \xrightarrow{x} K \to 0$ surjective $\Rightarrow K = 0$ by Nakayama and $\Rightarrow \varphi$ is an isomorphism $\square$

Lemma:

If R is a regular local ring, then any MCM is free.

Recall that if R CM, then any module M has a resolution:

$$0 \to K \to F_i \to \cdots \to F_0 \to M \to 0 \quad \text{(dimension shifting)}$$

with K MCM. ($i \leq d-1$). Thus we have an immediate corollary:

Cor. If R is regular local, then every finite R-module has a finite free resolution of length $\leq \dim R$. (More explicitly, we can find length ($\dim R - \operatorname{depth} M$) resolution). $\square$

Pf of lemma:

Assume M is MCM over R regular (CM).

$d=0$. $R=k$ and the result is trivial.

Let $x \in m$ be part of a minimal system of generators of m. Then x is a non-zero divisor on M (by the second lemma before, x is good w.r.t. the minimal system, thus non-zero divisor). Moreover, R/xR is regular of dimension 1 less and M/xM MCM over R/xR. Thus M/xM is free by induction and we are done by the previous lemma. □

Lemma: (Schanuel)

R: ring. M: an R-module. Suppose that $0 \to K \xrightarrow{i} P_1 \xrightarrow{p} M \to 0$, $0 \to L \xrightarrow{j} P_2 \xrightarrow{q} M \to 0$ are two s.e.s, and P_1, P_2 projective.

Then $K \oplus P_2 \cong L \oplus P_1$. ($K, L$ are stabely equivalent. If in particular, K is projective then L is also injective).

Pf: Consider

$$\begin{tikzcd}
0 \arrow[r] & \ker(p+q) \arrow[r] & P_1 \oplus P_2 \arrow[r, "p+q"] & M \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u] & P_1 \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Snake $\Rightarrow \ker(p+q)/K \cong P_2$. But P_2 projective $\Rightarrow \ker(p+q) \cong K \oplus P_2$. By symmetry $K \oplus P_2 \cong \ker(p+q) \cong L \oplus P_1$. □

Def. R: Noetherian local. A finite R module M has finite projective dimension if it has a finite length projective resolution, and $\operatorname{projdim}_R(M) \triangleq$ minimal length of such a resolution.

Lemma.

R: Noetherian local ring. M a finite R-module. Assume $\operatorname{projdim}_R(M) = d$. If $F_e \to F_{e-1} \to \cdots \to F_0 \to M \to 0$ is exact and F_i proj., $e \geq d-1$. Then $\ker(F_e \to F_{e-1})$ is finite projective.

□

Def. R: Noetherian. We say that R has finite global dimension if $\exists$ an integer n s.t. every finite R-module has a finite projective resolution of length $\leq n$. The minimal such n is called the global dimension of R.

Cor. A regular local ring has finite global dimension $\leq \dim R$.

Digression: What makes a complex exact?

Ref: Bushsbaum & Eisenbud, What makes a complex exact? J. Algebra 25 pp. 259-268. 1973.

Basic setting:

(R, m, k) Noetherian local ring.

Complex : $0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \to R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} R^{n_0} \to 0$ $(*)$

Lemma:

In the complex $(*)$, if any φ_i has a matrix coefficient a unit, then we can write $(*)$ up to isomorphism as a direct sum:

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \to R^{n_i - 1} \xrightarrow{\varphi_i} R^{n_{i-1}-1} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} R^{n_0} \to 0$$

$$\oplus$$

$$0 \to 0 \to 0 \to \cdots 0 \to R \xrightarrow{1} R \to 0 \to \cdots \to 0 \to 0$$

Pf: Simple linear algebra. $\square$

Complexes of the form $0 \to R \to R \to 0$ are called trivial.

Cor. Every complex of the form $(*)$ is isomorphic to

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \to R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} R^{n_0} \to 0$$

$$\oplus$$

Trivial ones.

and each φ_i has coefficients in m.

Lemma:

If R is Artinian, then any exact complex is a direct sum of trivial ones.

Pf: In view of the previous cor., it suffices to show that $\ker\varphi_e \neq 0$, where φ_e has its matrix coefficients in $\mathfrak{m}$.

But R Artinian $\Rightarrow \exists\, x \in R,\ x \neq 0,\ \mathfrak{m} \cdot x = 0 \Rightarrow 0 \neq \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \ker\varphi_e$. $\square$

Def. (Only for this digression) Given $\varphi: R^m \longrightarrow R^n$, we set

(1). $\operatorname{rank}(\varphi) = \max\{ r \mid \wedge^r\varphi : \wedge^r R^m \longrightarrow \wedge^r R^n \text{ is non-zero} \}$

(2). $I(\varphi) =$ ideal generated by determinants of $\operatorname{rank}\varphi \times \operatorname{rank}\varphi$ minors of φ.

Lemma: Suppose our complex

$$0 \longrightarrow R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \longrightarrow R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} R^{n_0} \longrightarrow 0$$

is a direct sum of trivial ones. Then:

(1). $rk(\varphi_i) \triangleq r_i = n_i - n_{i+1} + \cdots + (-1)^{e-i} n_e$

(2). $rk(\varphi_{i+1}) + rk(\varphi_i) = n_i$

(3). Each $I(\varphi_i) = R$.

Pf: Linear algebra. $\square$

Lemma: $\varphi: R^m \longrightarrow R^n$. TFAE:

(1). φ is injective

(2). $\operatorname{rank}(\varphi) = m$, and either $I(\varphi) = R$ or $I(\varphi)$ contains a non-zero divisor.

Pf: By the cor. we may assume φ has all coefficients in $\mathfrak{m}$.

Let $\mathfrak{q} \in \operatorname{Ass}(R)$, $x \in R$ with $\operatorname{ann}(x) = \mathfrak{q}$. Then

$$\begin{array}{ccccc} 0 & \longrightarrow & R^m & \xrightarrow{\varphi} & R^n \\ & & \uparrow & & \uparrow \\ 0 & \longrightarrow & (xR)^m & \xrightarrow{\varphi} & (xR)^m \end{array}$$

But $xR \cong R/\mathfrak{q}$. $\Rightarrow (R/\mathfrak{q})^m \overset{\varphi}{\hookrightarrow} (R/\mathfrak{q})^n \Rightarrow (\kappa(\mathfrak{q}))^m \hookrightarrow (\kappa(\mathfrak{q}))^n \Rightarrow$ some $m \times m$ minor of φ is not zero in $\kappa(\mathfrak{q}) \Rightarrow I(\varphi) \subsetneq \mathfrak{q}$. By prime avoidance, $\exists\, g \in I(\varphi)$, $g \notin \mathfrak{q}$, $\forall\, \mathfrak{q} \in \operatorname{Ass}(R) \Rightarrow g$ is a non-zero divisor.

Conversely, $I(\varphi) = R \Rightarrow \exists$ $m\times m$ minor invertible, thus φ must be injective.
Furthermore, notice that $\varphi: R^m \longrightarrow R^n$ is injective iff $\varphi_q: R_q^m \to R_q^n$ is injective $\forall q \in \operatorname{Ass} R$. (Indeed, "$\Rightarrow$" is easy; "$\Leftarrow$" $\forall x \in R^m$, $\varphi(x)=0 \Rightarrow \varphi_q(x) = 0 \Rightarrow x_q = 0$ $\Rightarrow \operatorname{ann} x \not\subseteq q$, $\forall q \in \operatorname{Ass} R$. By prime avoidance, $\exists$ r a non-zero divisor, $r \in \operatorname{ann} x \Rightarrow x = 0$.)
Now if $x \in I(\varphi)$ is a non-zero divisor, x becomes a unit in R_q, $\forall q \in \operatorname{Ass}(R)$ and it reduces to $I(\varphi) = R$ case. $\square$

Cor. If $m \in \operatorname{Ass} R$, then any exact complex is a direct sum of trivial ones. $\square$

Lemma. $x \in R$ a non-zero divisor.

$$0 \longrightarrow R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \longrightarrow R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} R^{n_0} \quad \text{exact}$$

$$\Rightarrow 0 \longrightarrow (R/xR)^{n_e} \longrightarrow \cdots \longrightarrow (R/xR)^{n_1} \quad (*)'$$

Pf: $0 \to (*) \longrightarrow (*) \longrightarrow (*)' \longrightarrow 0$ is a s.e.s. of complexes
$\Rightarrow$ (l.e.s) $(*)'$ is exact. $\square$

Lemma. (Acyclicity) R Noetherian local ring.
A complex $M_\bullet$: $0 \to M_e \to M_{e-1} \to \cdots \to M_0$ with $\operatorname{depth} M_i \geq i$.
Let i_0 be the largest index s.t. $M_\bullet$ is not exact at i.
If $i_0 > 0$, then $\operatorname{depth} H_{i_0}(M_\bullet) \geq 1$.
Pf: We break the complex into parts:

$$0 \to M_e \to M_{e-1} \to I_{e-1} \to 0$$
$$0 \to I_{e-1} \to M_{e-2} \to I_{e-2} \to 0$$
$$\vdots$$
$$0 \to I_{i_0+2} \to M_{i_0+1} \to I_{i_0+1} \to 0$$
$$0 \to I_{i_0+1} \to \ker d_{i_0} \to H_{i_0} \to 0$$

From a previous lemma, we know that
$\operatorname{depth} I_{e-1} \geq \min\{\operatorname{depth} M_{e-1}, \operatorname{depth} M_e - 1\} \geq e-1$
$\operatorname{depth} I_{e-2} \geq \min\{\operatorname{depth} M_{e-2}, \operatorname{depth} I_{e-1}\} \geq e-2$
$\cdots \Rightarrow \operatorname{depth} I_{i_0+1} \geq i_0+1$, and $\operatorname{depth} H_{i_0} \geq \min\{\operatorname{depth} \ker d_{i_0}, \operatorname{depth} I_{i_0+1} - 1\} \geq i_0 > 0$. $\square$

Thm. (R, m, k): Noetherian local ring. Consider the complex:

$$0 \longrightarrow R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \xrightarrow{\varphi_1} R^{n_0}$$

this complex is exact iff for all i, $1 \le i \le e$, we have:

(1). $\operatorname{rank}(\varphi_i) = r_i$ (as predicted from trivial cases)

(2). $I(\varphi_i) = R$ or $I(\varphi_i)$ contains an R-regular sequence of length i.

Pf: W.L.O.G. we may assume that the matrix coefficients of all φ_i are in m. Firstly we show that $(1)+(2) \Rightarrow$ exactness. For this purpose, we use induction on $\dim R$.

$\dim R = 0$, only $I(\varphi_i) = R$ could occur, then some matrix coefficients must be units, contradiction with our assumption unless the original complex is a direct sum of trivial ones.

Next, observe we may assume $e \geq 2$ by a previous lemma.

Claim: the complex localized at any prime $\mathfrak{p} \subseteq R$, $\mathfrak{p} \neq m$, is exact.

This is true since $\dim R_{\mathfrak{p}} < \dim R$ and conditions (1) and (2) still hold, thus the induction hypothesis applies.

It follows the cohomology groups $H_\cdot(M_\cdot)$ must be supported at $m \Rightarrow \dim H_\cdot(M_\cdot) = 0$

$(2) \Rightarrow \operatorname{depth} R \geq e$. Thus by the acyclic lemma, if any $H_\cdot(M_\cdot) \neq 0$, $\operatorname{depth} H_\cdot(M_\cdot) \geq 1$ which contradicts the fact that $\dim \geq \operatorname{depth}$.

Conversely, assume that the complex is exact. $\forall\, \mathfrak{q} \in \operatorname{Ass} R$, $\operatorname{depth} R_{\mathfrak{q}} = 0$ and the complex $\otimes R_{\mathfrak{q}}$ is exact. By a previous lemma, the localized complex must be a direct sum of trivial ones. Thus the ranks must then match $I(\varphi_i) = R_{\mathfrak{q}}$, $\forall i$. Hence $I(\varphi_i) \subsetneq \mathfrak{q}$, $\forall i$. $\forall \mathfrak{q} \in \operatorname{Ass} R \Rightarrow \prod_{i=1}^{e} I(\varphi_i) \subsetneq \mathfrak{q}$, $\forall \mathfrak{q} \Rightarrow \prod_{i=1}^{e} I(\varphi_i)$ contains a non-zero divisor in R, by prime avoidance.

By a previous lemma, $0 \to (R/x)^{n_e} \to (R/x)^{n_{e-1}} \to \cdots (R/x)^{n_1}$ is still exact.

By induction on e ($e=1$ is a lemma), $I(\varphi_i)/xR \subseteq R/xR$ have depth $\geq i-1$ and we are done. $\square$

"How cool that is!"

Application:

Lemma:

Let R be a Noetherian local ring. Suppose the residue field κ has finite projective dimension n over R. Then $\operatorname{depth} R \geq n$

Rk: for many rings. $\operatorname{projdim}_R \kappa = +\infty$.

Pf: It suffices to prove for $\operatorname{projdim}_R \kappa = n < \infty$, i.e. $\exists$ an exact free complex

$$0 \to P_n \xrightarrow{\varphi_n} P_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} P_0 \to \kappa \to 0$$

We may take the complex minimal so that the matrix coefficients of the boundary maps are in m. Thm $\Rightarrow I(\varphi_1) \subseteq m$ contains a regular sequence of length n

$\square$

Lemma.

Suppose R is Noetherian local with maximal ideal m, residue field κ. Then $\operatorname{Projdim}_R(\kappa) \geq \dim_\kappa m/m^2$.

Rk: note that we always have $\dim R \leq \dim_\kappa m/m^2$, and for regular local rings $\operatorname{projdim}_R(\kappa) \leq \dim R = \operatorname{depth} R \Rightarrow \dim R = \dim_\kappa m/m^2 = \operatorname{projdim}_R(\kappa)$.

Pf: Pick any finite free resolution of κ, with all matrix coefficients in m.

$$F_\bullet: \ 0 \to F_e \to F_{e-1} \to \cdots \to F_1 \to F_0 \to \kappa \to 0.$$

Pick $x_1, \cdots, x_n \in m$ that form a basis of m/m^2. Form the Koszul complex.

$$K_\bullet(R, x_i): \ 0 \to \Lambda^n R^n \to \Lambda^{n-1} R^n \to \cdots \to \Lambda^2 R^n \to R^n \to R$$

with boundary maps: $e_{j_1} \wedge \cdots \wedge e_{j_a} \wedge \cdots \wedge e_{j_k} \mapsto \sum_{a=1}^{k} (-1)^{a-1} x_{j_a} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_a}} \wedge \cdots \wedge e_{j_k}$.

Clearly, $H_0(K_\bullet(R, x_i)) = R/(x_1, \cdots, x_n) = R/m = \kappa$. Since $F_\bullet$ is exact, and $K_\bullet(R, x_i)$ is a complex, a previous lemma $\Rightarrow \exists$ a map of complexes inducing $\mathrm{id}_{H_0} = \mathrm{id}_\kappa$:

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda^e R^n \arrow[r] \arrow[d, "\alpha_e"] & \Lambda^{e-1} R^n \arrow[r] \arrow[d, "\alpha_{e-1}"] & \cdots \arrow[r] & R^n \arrow[r] \arrow[d, "\alpha_1"] & R \arrow[r] \arrow[d, "\alpha_0"] & \kappa \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & F_e \arrow[r] & F_{e-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & \kappa \arrow[r] & 0
\end{tikzcd}$$

Observe that F_0 can be taken to be R. Then α_0 is necessarily multiplication by a unit.

Claim: $\alpha_1 \otimes \kappa$ is injective.

Indeed:

$$\begin{tikzcd} R^n \otimes \kappa \arrow[r, "\cong"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\circlearrowleft"] & m \otimes \kappa \arrow[d, "\substack{\text{multiplication}\\ \text{by a unit}}"] \\ F_1 \otimes \kappa \arrow[r, "\cong"] & m \otimes \kappa \end{tikzcd}$$

$\Rightarrow$ $\alpha_1 \otimes id_\kappa$ is injective

Inductively, we shall show that $\alpha_i \otimes \kappa$ is injective. $\forall i$

$$\begin{tikzcd} \wedge^i \kappa^n \cong \wedge^i R^n \otimes m \arrow[r, "\partial_i"] \arrow[d, "\alpha_i \otimes \kappa"] & \wedge^{i-1}(mR^n) \otimes \kappa \cong \wedge^{i-1}(R^n) \otimes \kappa^n \arrow[d, "\alpha_{i-1} \otimes \kappa^n"] \\ F_i \otimes m \arrow[r] & mF_{i-1} \otimes \kappa \end{tikzcd} \quad (F_i \to mF_{i-1}, \text{ by assumption})$$

But by explict computation, one can show that ∂_i is injective and by induction hypothesis $\alpha_{i-1} \otimes \kappa^n$ is injective $\Rightarrow$ $\alpha_i \otimes \kappa$ is injective.

Hence we conclude that $e \geq n$ and we are done. $\square$

Combining the above results, we obtain:

Thm. (Characterization of Regular Local Rings)

Let R be a Noetherian local ring. TFAE:

(1). R is regular.

(2). κ has finite projective dimension.

(3). R has finite global dimension.

(4). For every finite R-module M, we have $\mathrm{projdim}_R(M) + \mathrm{depth}_R(M) = \dim R$.

Pf: (4) $\Rightarrow$ (3) $\Rightarrow$ (2) is trivial

(2) $\Rightarrow$ (1) If $\mathrm{projdim}_R \kappa < \infty$. $\dim R \geq \mathrm{depth} R \geq \mathrm{projdim}_R \kappa \geq \dim_\kappa m/m^2 \geq \dim R$.

(1) $\Rightarrow$ (4) Recall that R regular local. of dim n (CM, in particular), M finite over R of depth e, then M has a finite free resolution: (dim shifting + MCM over regular local rings are free):

$$0 \to F_{n-e} \xrightarrow{\varphi_{n-e}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$$

Note that this is a minimal resolution since, again by dimension shifting, any minimal resolution has $\dim \ker \varphi_i = n+i$. $\square$

Lemma:

Suppose R has finite global dimension, then $S^{-1}R$ has finite global dimension, $\forall$ multiplicatively closed set.

Pf: Let N be any finite $S^{-1}R$ module, generated by $n_1, \cdots, n_r$ over $S^{-1}R$. Let M be the R-submodule generated by these n_i's inside N. Then M has a finite projective resolution: $0 \to P_\bullet \to M \to 0$.

Localizing $(\otimes_R S^{-1}R)$: $0 \to S^{-1}P_\bullet \to S^{-1}M = N \to 0$ gives a projective resolution of N as $S^{-1}R$-module. $\square$

Cor. R regular local $\Rightarrow R_{\mathfrak{p}}$ regular local. $\square$

Def. A Noetherian ring is called regular iff all of its local rings are regular.

. A regular local ring can have infinite dimension.

Cor. R Noetherian. R has finite global dimension $\Leftrightarrow$ R is regular and $\dim R < \infty$.

Pf: "$\Rightarrow$" by the lemma, $R_{\mathfrak{p}}$ has finite global dimension, $\forall \mathfrak{p}$. Thus $R_{\mathfrak{p}}$ regular.

"$\Leftarrow$" $\forall$ R-module M. Take a free resolution of length $\dim R - 1$:

$$0 \to K \to F_{d-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

Now at every prime $\mathfrak{p}$, $K_{\mathfrak{p}}$ is free since $R_{\mathfrak{p}}$ regular with $\dim R_{\mathfrak{p}} \le \dim R$. It followes that K is projective. $\square$

§12. Good Properties of Polynomial Rings

Later we shall study varieties (schemes).

$\mathcal{Aff}$: the category of affine schemes (varieties) $\xleftrightarrow{\text{opp}}$ $\mathcal{Alg}$: the category of finite algebras over k (domains)

Via glueing, we will construct arbitrary schemes (varieties). Thus we first study some good properties of polynomial rings. (good properties of affine schemes, very primitive).

Dimension of finite type algebras over fields.

Lemma:

Let $m \subseteq k[x_1, \cdots, x_n]$ be a maximal ideal. then m is generated by n elements.

Pf: By Hilbert Nullstellensatz, $\kappa(m)$ is a finite extension of k. We prove the result by induction. Let $m' = m \cap k[x_1, \cdots, x_{n-1}]$, then $k \subseteq k[x_1, \cdots, x_{n-1}]/m' \subseteq \kappa(m) \Rightarrow m'$ is maximal. By induction hypothesis, $m' = (f_1(x_1), \cdots, f_{n-1}(x_1, \cdots, x_{n-1}))$ $\Rightarrow \kappa(m) = \kappa(m')[x_n]/(f_n(x_n))$ for some $f_n(x_n) \in \kappa(m')[x_n]$, which can be taken to be a polynomial in $k[x_1, \cdots, x_n]$. $\square$

Lemma.

Assumption as above, then $\dim k[x_1, \cdots, x_n]_m = n$.

Pf: $0 \subseteq (f_1) \subseteq (f_1, f_2) \subseteq \cdots \subseteq (f_1, \cdots, f_n) = m$ is a chain of primes of length n. But $(f_1, \cdots, f_n)$ generates $m \Rightarrow \dim k[x_1, \cdots, x_n]_m = n$. $\square$

Cor. (1). $k[x_1, \cdots, x_n]_m$ is regular local of dim n.

(2). $k[x_1, \cdots, x_n]$ has finite global dimension n. $\square$

Note that in the previous section. to show (1) $\Rightarrow$ (2), we used that any locally free module is projective. By a conjecture (theorem now) of Serre states that any finite projective module over $k[x_1, \cdots, x_n]$ is actually free.

Properties:

1). Given a prime ideal $\mathfrak{p} \subseteq k[x_1,\cdots,x_n]$, then $k[x_1,\cdots,x_n]/\mathfrak{p}$ is equidimensional of dimension $n-ht\,\mathfrak{p}$.

Pf: We did the proof before for the case $k=\bar{k}$. Now, with the aid of the previous cor, we may remove the requirement "$k=\bar{k}$."

Suppose $0=\bar{\mathfrak{p}}_0 \subseteq \bar{\mathfrak{p}}_1 \subseteq \cdots \subseteq \bar{\mathfrak{p}}_r$ be a maximal chain of primes, let $\mathfrak{p}_i$ be the preimages in $k[x_1,\cdots,x_n]$, then $\mathfrak{p}_r$ is necessarily maximal and $\mathfrak{p}_0=\mathfrak{p}$.

Furthermore, $k[x_1,\cdots,x_n]_{\mathfrak{p}}$ is regular local $\Rightarrow$ every maximal chain of primes descending from $\mathfrak{p}$ has length $ht\,\mathfrak{p}$. Combined, these chains give rise to a maximal chain of primes in $k[x_1,\cdots,x_n]$ (or $k[x_1,\cdots,x_n]_{\mathfrak{m}}$) $\Rightarrow$ $ht\,\mathfrak{p}+\dim R/\mathfrak{p}=n$. $\square$

2). $\forall\, \mathfrak{p} \subseteq \mathfrak{q} \subseteq k[x_1,\cdots,x_n]$ primes. Then every maximal chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length $ht\,\mathfrak{q}-ht\,\mathfrak{p}$.

Pf: k is u.c. $\square$

3). $\dim k[x_1,\cdots,x_n]/\mathfrak{p} = tr.\deg_k f.f.(k[\underline{x}]/\mathfrak{p})$.

We shall prove the result using the next:

Lemma 1: k a field, $(0) \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}$ a maximal chain of primes in $k[x_1,\cdots,x_n]$. Then $\exists\; k[y_1,\cdots,y_n] \hookrightarrow k[x_1,\cdots,x_n]$ finite s.t. $\mathfrak{p}_i \cap k[y_1,\cdots,y_n]=(y_1,\cdots y_i)$.

Proof of 3).

Say $\mathfrak{p}=\mathfrak{p}_i$ in a maximal chain of primes as in Lemma 1. Then

$$k[y_{i+1},\cdots,y_n]=k[y_1,\cdots,y_n]/(y_1,\cdots,y_i) \hookrightarrow k[\underline{x}]/\mathfrak{p}$$

and the map is finite by lemma 1.

$\Rightarrow k(y_{i+1},\cdots,y_n) \hookrightarrow f.f.(k[x]/\mathfrak{p})$ is finite

$\Rightarrow tr.\deg_k k[\underline{x}]/\mathfrak{p} = n-i = n-ht\,\mathfrak{p} = \dim k[\underline{x}]/\mathfrak{p}$. $\square$

Lemma 2. Given $f \in k[\underline{x}]$, non-constant, then $\exists\, \varphi: k[\underline{y}] \longrightarrow k[\underline{x}]$ finite, s.t $\varphi(y_1)=f$ and φ is an injection.

Pf: Write $f = \sum_I a_I \underline{x}^I$ (some $a_I \neq 0$, $I \neq 0$, by assumption). W.L.O.G. we may assume that x_1 occurs in f. Consider

$$f(x_1, y_2+x_1^{e_2}, \cdots, y_n+x_1^{e_n}) = \sum_{I=(i_1,\cdots,i_n)} (a_I x_1^{i_1+i_2e_2+\cdots+i_ne_n} + \text{l.o.t in } x_1)$$

the l.o.t. has coefficients polynomials in $y_2,\cdots,y_n$. Now pick $1 \ll e_1 \ll e_2 \ll \cdots \ll e_n$ so that all integers $i_1+i_2e_2+\cdots+i_ne_n$ occuring are pairwise distinct. Hence:

$$f = c \cdot x_1^N + \text{l.o.t in } x_1, \quad c \in k.$$

Set $\varphi: y_1 \mapsto f, \quad y_2 \mapsto x_2 - x_1^{e_1}, \cdots, y_n \mapsto x_n - x_1^{e_n}$.

Claim: x_1 generates $k[\underline{x}]$ over $k[\underline{y}]$ as a ring, and x_1 satisfies a monic polynomial over $k[\underline{y}]$.

Indeed, $\varphi(y_1) = f_1(x_1,\cdots,x_n) = f_1(x_1, \varphi(y_2)+x_1^{e_1}, \cdots, \varphi(y_n)+x_1^{e_n})$

$$= c \cdot x_1^N + \text{l.o.t. in } x_1$$

$\Rightarrow c x_1^N + \cdots + c_N - \varphi(y_1) = 0$ is a monic equation.

That φ is an injection is easy by def. of φ. □

Rmk: A more general fact is that: $R \to S$, S finitely generated by $x_1,\cdots,x_r$ over R, each x_i satisfies a monic polynomial over R, then S/R is finite.

Lemma 3. If $R \to S$ is a finite ring map, and q, q' are distinct primes in S lying over the same prime, then neither $q' \subseteq q$ nor $q \subseteq q'$.

Pf: Recall that primes lying over $\mathfrak{p} \overset{1:1}{\longleftrightarrow}$ primes in $S \otimes_R \kappa(\mathfrak{p})$. S finite over $R \Rightarrow S \otimes_R \kappa(\mathfrak{p})$ is finite over $\kappa(\mathfrak{p})$, thus Artinian, and the distinct primes are all maximal, thus incomparable. □

Proof of lemma 1

Pick $f \in \mathfrak{p}_1$, not a constant. Lemma 2 $\Rightarrow \exists\, k[y_1, z_2, \cdots z_n] \overset{\varphi}{\hookrightarrow} k[\underline{x}]$ finite such that $\varphi(y_1) = f \in \mathfrak{p}_1$. By lemma 3, all the prime ideals $(0) \cap k[y_1, z_2, \cdots z_n] \subsetneq k[y_1, z_2, \cdots z_n] \cap \mathfrak{p}_1 \subsetneq \cdots \subsetneq k[y_1, z_2, \cdots, z_n] \cap \mathfrak{p}_n$. Since $k[y_1, z_2, \cdots z_n]$ is catenary, and has dim $n \Rightarrow k[y_1, z_2, \cdots z_n] \cap \mathfrak{p}_1 = (z_1)$. (Note that this also forces $(0) \cap k[y_1, z_2, \cdots z_n] = (0)$ and φ is injective).

Inductively, assume we have $k[y_1, \cdots, y_i, z_{i+1}, \cdots z_n] \hookrightarrow k[x_1, \cdots, x_n]$ finite and $0 \subsetneq \mathfrak{p}_i \cap k[y_1, \cdots y_i, z_{i+1}, \cdots, z_n] = (y_1, \cdots, y_i) \subsetneq \cdots \subsetneq \mathfrak{p}_i \cap k[y_1, \cdots y_i, z_{i+1}, \cdots z_n] \subsetneq$

$\mathfrak{p}_{i+1} \cap k[y_1,\cdots,y_i,z_{i+1},\cdots z_n] \subsetneq \cdots \subsetneq \mathfrak{p}_n \cap k[y_1,\cdots y_i,z_{i+1},\cdots z_n]$.

Choose any $f \in \mathfrak{p}_{i+1} \cap k[y_1,\cdots,y_i,z_{i+1},\cdots z_i]$, $f \notin \mathfrak{p}_i \cap k[y_1,\cdots,y_i,z_{i+1},\cdots z_i]$, we may write $f = g(z_{i+1},\cdots,z_n) + \sum_{j=1}^{i} y_j h_j$. Then $g \notin \mathfrak{p}_i \cap k[y_1,\cdots y_i,z_{i+1},\cdots,z_i] = (y_1,\cdots,y_i)$. Lemma 2 $\Rightarrow \exists$ a finite map $\psi_0: k[y_{i+1},z'_{i+2},\cdots,z'_n] \longrightarrow k[z_{i+1},\cdots,z_n]$ with $\psi_0(y_{i+1}) = g$. Take the base change, we obtain $\psi: k[y_1,\cdots,y_i,y_{i+1},z'_{i+2}\cdots z'_n] \to k[y_1,\cdots,y_i,z_i,\cdots z_n]$ still a finite map. (Base change preserves finiteness). Composing with the original map $k[y_1,\cdots,y_i,z_{i+1},\cdots z_n] \hookrightarrow k[\underline{X}]$, we obtain a finite map $k[y_1,\cdots,y_{i+1},z'_{i+2}\cdots z'_n] \to k[\underline{X}]$. Moreover, $\mathfrak{p}_k \cap k[y_1,\cdots,y_{i+1},z'_{i+2}\cdots z'_n] = (\mathfrak{p}_k \cap k[y_1,\cdots y_i,z_{i+1},\cdots z_n]) \cap k[y_1,\cdots,y_{i+1},z'_{i+2},\cdots z'_n]$ $= (y_1,\cdots,y_k) \cap k[y_1,\cdots,y_{i+1},z'_{i+2},\cdots z'_n] = (y_1,\cdots,y_k)$ for $0 \le k \le i$ (which also forces the composite map to be injective again), and for $k = i+1$:

$$\begin{aligned}\mathfrak{p}_{i+1} \cap k[y_1,\cdots,y_{i+1},z'_{i+2},\cdots z'_n] &= (\mathfrak{p}_{i+1} \cap k[y_1,\cdots,y_i,z_{i+1},\cdots,z_n]) \cap k[y_1,\cdots,y_{i+1},z'_{i+2},\cdots z'_n] \\ &\supseteq (g,y_1,\cdots,y_i) \cap k[y_1,\cdots,y_{i+1},z'_{i+2},\cdots z'_n] \\ &\supseteq (y_1,\cdots,y_{i+1})\end{aligned}$$

Again by the catenary property of $k[y_1,\cdots,y_{i+1},z'_{i+2}\cdots,z'_n]$, we have

$$\mathfrak{p}_{i+1} \cap k[y_1,\cdots,y_{i+1},z'_{i+2},\cdots z'_n] = (y_1,\cdots,y_{i+1})$$

and we are done. $\square$