Commutative Algebra

Excercises 1

The Spectrum of a ring [Reference: EGA I, Chapter 0 §2]

-2. Compute $\operatorname{Spec}\mathbb{Z}$ as a set and describe its topology.

Let A be any ring. Let X be any topological space.

-1. For $f \in A$ we define $D(f) := \operatorname{Spec} A \setminus V(f)$. Prove that the open subsets $D(f)$ form a basis of the topology of $\operatorname{Spec} A$.

0. Prove that the map $I \mapsto V(I)$ defines a natural bijection

$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset \operatorname{Spec} A \text{ closed}\}$$

A topological space X is called *quasi-compact* if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $X = U_{i_1} \cup \ldots U_{i_n}$.

1. Prove that $\operatorname{Spec} A$ is quasi-compact for any ring A.

A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ there is an open subset of X containing one but not the other. We say that X is *Hausdorff* if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

2. Show that $\operatorname{Spec} A$ is **not** Hausdorff in general. Prove that $\operatorname{Spec} A$ is T_0. Give an example of a topological space X that is not T_0.

Remark: usually the word compact is reserved for quasi-compact and Hausdorff spaces. A topological space X is called *irreducible* if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called irreducible if it is an irreducible topological space with the topology induced from X. This definitions implies T is irreducible if and only if the closure $\bar{T}$ of T in X is irreducible.

3. Prove that $\operatorname{Spec} A$ is irreducible if and only if $Nil(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A.

4. Prove that a closed subset $T \subset \operatorname{Spec} A$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$.

5. Show that in a T_0 space X every irreducible closed subset has at most one generic point.

6. Prove that in $\operatorname{Spec} A$ every irreducible closed subset *does* have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of $\operatorname{Spec} A$ with the set of irreducible closed subsets of X.

7. Give an example to show that an irreducible subset of $\operatorname{Spec}\mathbb{Z}$ does not neccesarily have a generic point.

A topological space X is called *Noetherian* if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \dots$ of closed subsets of X stabilizes. (It is called *Artinian* if any increasing sequence of closed subsets stabilizes.)

8. Show that if the ring A is Noetherian then the topological space $\operatorname{Spec} A$ is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

A maximal irreducible subset $T \subset X$ is called an *irreducible component* of the space X. Such an irreducible component of X is automatically a closed subset of X.

9. Prove that any irreducible subset of X is contained in an irreducible component of X.

10. Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \dots, X_n$, and that $X = X_1 \cup X_2 \cup \dots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbb{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets. This is not terribly interesting.)

11. Show that irreducible components of $\operatorname{Spec} A$ correspond to minimal primes of A.

A point $x \in X$ is called closed if $\overline{\{x\}} = \{x\}$. Let x, y be points of X. We say that x is a *specialization* of y, or that y is a *generalization* of x if $x \in \overline{\{y\}}$.

12. Show that closed points of $\operatorname{Spec} A$ correspond to maximal ideals of A.

13. Show that $\mathfrak{p}$ is a generalization of $\mathfrak{q}$ in $\operatorname{Spec} A$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space X is called a generic point if it is a generic points of one of the irreducible components of X.)

14. Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

A topological space X is called *connected* if it is not the union of two nonempty disjoint open subsets. A *connected component* of X is a (nonempty) maximal connected subset. Any point of X is contained in a connected component of X and any connected component of X is closed in X. (But in general a connected component need not be open in X.)

15. Show that $\operatorname{Spec} A$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C.

16. Let T be a connected component of $\operatorname{Spec} A$. Prove that T is stable under generalization. Prove that T is an open subset of $\operatorname{Spec} A$ if A is Noetherian. (Remark: This is wrong when A is an infinite product of copies of $\mathbb{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

17. Compute $\operatorname{Spec} k[x]$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when k is algebraically closed but also when k is not.)

18. Compute $\operatorname{Spec} k[x, y]$, where k is algebraically closed. [Hint: use the morphism $\varphi : \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x]$; if $\varphi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think algebraic geometers call this affine 2-space?)

19. Compute $\operatorname{Spec} \mathbb{Z}[y]$. [Hint: as above.] (Affine 1-space over $\mathbb{Z}$.)