Commutative Algebra

Exercises 1

The Spectrum of a ring [Reference: EGA I, Chapter 0 §2]

-2. Compute Spec \( \mathbb{Z} \) as a set and describe its topology.

Let \( A \) be any ring. Let \( X \) be any topological space.

-1. For \( f \in A \) we define \( D(f) := \text{Spec } A \setminus V(f) \). Prove that the open subsets \( D(f) \) form a basis of the topology of \( \text{Spec } A \).

0. Prove that the map \( I \mapsto V(I) \) defines a natural bijection \( \{ I \subset A \text{ with } I = \sqrt{I} \} \rightarrow \{ T \subset \text{Spec } A \text{ closed} \} \).

A topological space \( X \) is called quasi-compact if for any open covering \( X = \bigcup_{i \in I} U_i \) there is a finite subset \( \{ i_1, \ldots, i_n \} \subset I \) such that \( X = U_{i_1} \cup \ldots \cup U_{i_n} \).

1. Prove that Spec \( A \) is quasi-compact for any ring \( A \).

A topological space \( X \) is said to verify the separation axiom \( T_0 \) if for any pair of points \( x, y \in X, x \neq y \) there is an open subset of \( X \) containing one but not the other. We say that \( X \) is Hausdorff if for any pair \( x, y \in X, x \neq y \) there are disjoint open subsets \( U, V \) such that \( x \in U \) and \( y \in V \).

2. Show that Spec \( A \) is not Hausdorff in general. Prove that Spec \( A \) is \( T_0 \). Give an example of a topological space \( X \) that is not \( T_0 \).

Remark: usually the word compact is reserved for quasi-compact and Hausdorff spaces. A topological space \( X \) is called irreducible if \( X \) is not empty and if \( X = Z_1 \cup Z_2 \) with \( Z_1, Z_2 \subset X \) closed, then either \( Z_1 = X \) or \( Z_2 = X \). A subset \( T \subset X \) of a topological space is called irreducible if it is an irreducible topological space with the topology induced from \( X \). This definitions implies \( T \) is irreducible if and only if the closure \( \bar{T} \) of \( T \) in \( X \) is irreducible.

3. Prove that Spec \( A \) is irreducible if and only if \( \text{Nil}(A) \) is a prime ideal and that in this case it is the unique minimal prime ideal of \( A \).

4. Prove that a closed subset \( T \subset \text{Spec } A \) is irreducible if and only if it is of the form \( T = V(p) \) for some prime ideal \( p \subset A \).

A point \( x \) of an irreducible topological space \( X \) is called a generic point of \( X \) if \( X \) is equal to the closure of the subset \( \{ x \} \).

5. Show that in a \( T_0 \) space \( X \) every irreducible closed subset has at most one generic point.

6. Prove that in Spec \( A \) every irreducible closed subset does have a generic point. In fact show that the map \( p \mapsto \{ p \} \) is a bijection of Spec \( A \) with the set of irreducible closed subsets of \( X \).

7. Give an example to show that an irreducible subset of Spec \( \mathbb{Z} \) does not neccesarily have a generic point.
A topological space $X$ is called \textit{Noetherian} if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of $X$ stabilizes. (It is called \textit{Artinian} if any increasing sequence of closed subsets stabilizes.)

8. Show that if the ring $A$ is Noetherian then the topological space $\text{Spec } A$ is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

A maximal irreducible subset $T \subset X$ is called an \textit{irreducible component} of the space $X$. Such an irreducible component of $X$ is automatically a closed subset of $X$.

9. Prove that any irreducible subset of $X$ is contained in an irreducible component of $X$.

10. Prove that a Noetherian topological space $X$ has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any $X$ is always the union of its irreducible components, but that if $X = \mathbb{R}$ with its usual topology for instance then the irreducible components of $X$ are the one point subsets. This is not terribly interesting.)

11. Show that irreducible components of $\text{Spec } A$ correspond to minimal primes of $A$.

A point $x \in X$ is called closed if $\{x\} = \overline{\{x\}}$. Let $x, y$ be points of $X$. We say that $x$ is a \textit{specialization} of $y$, or that $y$ is a \textit{generalization} of $x$ if $x \in \overline{\{y\}}$.

12. Show that closed points of $\text{Spec } A$ correspond to maximal ideals of $A$.

13. Show that a generalization $p$ of $q$ in $\text{Spec } A$ if and only if $p \subset q$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space $X$ is called a generic point if it is a generic points of one of the irreducible components of $X$.)

14. Let $I$ and $J$ be ideals of $A$. What is the condition for $V(I)$ and $V(J)$ to be disjoint? A topological space $X$ is called \textit{connected} if it is not the union of two nonempty disjoint open subsets. A \textit{connected component} of $X$ is a (nonempty) maximal connected subset. Any point of $X$ is contained in a connected component of $X$ and any connected component of $X$ is closed in $X$. (But in general a connected component need not be open in $X$.)

15. Show that $\text{Spec } A$ is disconnected iff $A \cong B \times C$ for certain nonzero rings $B, C$.

16. Let $T$ be a connected component of $\text{Spec } A$. Prove that $T$ is stable under generalization. Prove that $T$ is an open subset of $\text{Spec } A$ if $A$ is Noetherian. (Remark: This is wrong when $A$ is an infinite product of copies of $\mathbb{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

17. Compute $\text{Spec } k[x]$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when $k$ is algebraically closed but also when $k$ is not.)

18. Compute $\text{Spec } k[x, y]$, where $k$ is algebraically closed. [Hint: use the morphism $\varphi : \text{Spec } k[x, y] \to \text{Spec } k[x]$; if $\varphi(p) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think algebraic geometers call this affine 2-space?)

19. Compute $\text{Spec } \mathbb{Z}[y]$. [Hint: as above.] (Affine 1-space over $\mathbb{Z}$.)