Commutative Algebra

Exercises 10

A Noetherian ring A is said to be *catenary* if for any triple of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ we have

$$ht(\mathfrak{p}_3/\mathfrak{p}_1) = ht(\mathfrak{p}_3/\mathfrak{p}_2) + ht(\mathfrak{p}_2/\mathfrak{p}_1).$$

1. Show that a Noetherian local domain of dimension 2 is catenary.

2. Let A be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X-1)$. Show that

$$A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}.$$

3. Find an example of a non Noetherian ring R such that every finitely generated ideal of R is finitely presented as an R-module. (A ring is said to be *coherent* if the last property holds.)

4. Consider the domain

$$\mathbb{Q}[r, s, t]/(s^2 - (r-1)(r-2)(r-3), t^2 - (r+1)(r+2)(r+3)).$$

Find a domain of the form $\mathbb{Q}[x, y]/(f)$ with isomorphic field of fractions.

5. Give an example of a finite type $\mathbb{Z}$-algebra R with the following two properties:
(a) There is no ring map $R \to \mathbb{Q}$.
(b) For every prime p there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} =\cong \mathbb{F}_p$.

6. For $f \in \mathbb{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbb{F}_p[x]$. Give an example of an $f \in \mathbb{Z}[x, u]$ such that the following two properties hold:
(a) There exist infinitely many p such that f_p does not have a zero in $\mathbb{F}_p$.
(b) For all $p >> 0$ the polynomial f_p either has a linear or a quadratic factor.

7. For $f \in \mathbb{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \bmod p \in \mathbb{F}_p[x, y]$. Give an "interesting" example of an f such that f_p is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^p y = xy(x^{p-1} - y^{p-1})$ is "uninteresting"; any f depeding only on x, u is "uninteresting", etc.