A Noetherian ring $A$ is said to be *catenary* if for any triple of prime ideals $p_1 \subset p_2 \subset p_3$ we have 
\[ ht(p_3/p_1) = ht(p_3/p_2) + ht(p_2/p_1). \]

1. Show that a Noetherian local domain of dimension 2 is catenary.

2. Let $A$ be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X - 1)$. Show that 
\[ A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}. \]

3. Find an example of a non Noetherian ring $R$ such that every finitely generated ideal of $R$ is finitely presented as an $R$-module. (A ring is said to be *coherent* if the last property holds.)

4. Consider the domain 
\[ \mathbb{Q}[r, s, t]/(s^2 - (r - 1)(r - 2)(r - 3), t^2 - (r + 1)(r + 2)(r + 3)). \]
Find a domain of the form $\mathbb{Q}[x, y]/(f)$ with isomorphic field of fractions.

5. Give an example of a finite type $\mathbb{Z}$-algebra $R$ with the following two properties:
   (a) There is no ring map $R \to \mathbb{Q}$.
   (b) For every prime $p$ there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} = \cong \mathbb{F}_p$.

6. For $f \in \mathbb{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \mod p \in \mathbb{F}_p[x]$. Give an example of an $f$ in $\mathbb{Z}[x, u]$ such that the following two properties hold:
   (a) There exist infinitely many $p$ such that $f_p$ does not have a zero in $\mathbb{F}_p$.
   (b) For all $p >> 0$ the polynomial $f_p$ either has a linear or a quadratic factor.

7. For $f \in \mathbb{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \mod p \in \mathbb{F}_p[x, y]$. Give an “interesting” example of an $f$ such that $f_p$ is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^py = xy(x^{p-1} - y^{p-1})$ is “uninteresting”; any $f$ depending only on $x, u$ is “uninteresting”, etc.