Commutative Algebra

Excercises 2

1. Let $(I, \geq)$ be a partially ordered set which is directed. Let A be a ring and let $(N_i, \varphi_{i,i'})$ be a directed system of A-modules indexed by I. Suppose that M is another A-module. Prove that

$$\lim_{i \in I} M \otimes_A N_i \cong M \otimes_A \left(\lim_{i \in I} N_i \right).$$

Remark. A module M over R is said to be of finite presentation over R if it is isomorphic to the cokernel of a map of finite free modules $R^{\oplus n} \to R^{\oplus m}$.

2. Prove that any module over any ring is
a. the limit of its finitely generated submodules, and
b. in some way a limit of finitely presented modules.

3. Let S be a multiplicative subset of the ring A.
a. For an A-module M show that $S^{-1}M = S^{-1}A \otimes_A M$.
b. Show that $S^{-1}A$ is flat over A.

4. Find an injection $M_1 \to M_2$ of A-modules such that $M_1 \otimes N \to M_2 \otimes N$ is not injective in the following cases:
a. $A = k[x, y]$ and $N = (x, y) \subset A$. (Here and below k is a field.)
b. $A = k[x, y]$ and $N = A/(x, y)$.

5. Give an example of a ring A and a finite A-module M which is a flat but not a projective A-module.

Remark. If M is of finite presentation and flat over A, then M is projective over A. Thus your example will have to involve a ring A which is not Noetherian. I know of an example where A is the ring of $\mathcal{C}^\infty$-functions on $\mathbb{R}$.

6. Let $A = k[x, y]_{(x,y)}$ be the local ring of the affine plane at the origin. Make any assumption you like about the field k. Suppose that $f = x^3 + x^2y^2 + y^{100}$ and $g = y^3 - x^{999}$. What is the length of $A/(f, g)$ as an A-module? (Possible way to proceed: think about the ideal that f and g generate in quotients of the form $A/\mathfrak{m}_A^n = k[x, y]/(x, y)^n$ for varying n. Try to find n such that $A/(f, g) + \mathfrak{m}_A^n \cong A/(f, g) + \mathfrak{m}_A^{n+1}$ and use NAK.)