Commutative Algebra

Excercises 3

1. Find a flat but not free module over $\mathbb{Z}_{(2)}$.

2. Let k be any field. Suppose that $A = k[[x,y]]/(f)$ and $B = k[[u,v]]/(g)$, where $f = xy$ and $g = uv + \delta$ with $\delta \in (u,v)^3$. Show that A and B are isomorphic rings.

Remark. A singularity on a curve over a field k is called an ordinary double point if the complete local ring of the curve at the point is of the form $k'[[x,y]]/(f)$, where (a) k' is a finite separable extension of k, (b) the initial term of f has degree two, i.e., it looks like $q = ax^2 + bxy + cy^2$ for some $a, b, c \in k'$ not all zero, and (c) q is a nondegenerate quadratic form over k' (in char 2 this means that b is not zero). In general there is one isomorphism class of such rings for each isomophism class of pairs (k', q).

3. Suppose that A is a ring and M is an A-module. Let f_i be a collection of elements of A such that

$$\mathrm{Spec}(A) = \bigcup D(f_i).$$

(a) Show that if M_{f_i} is a finitely generated A_{f_i}-module, then M is a finitely generated A-module.

(b) Show that if M_{f_i} is a flat A_{f_i}-module, then M is a flat A-module. (This is kind of silly if you think about it right.)

Remark. In algebraic geometric language this means that the property of "being finitely generated" or "being flat" is local for the Zariski topology (in a suitable sense). You can also show this for the property "being of finite presentation".

4. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring. For any finite A-module M define $r(M)$ to be the minimum number of generators of M as an A-module. This number equals $\dim_k M/\mathfrak{m}M = \dim_k M \otimes_A k$ by NAK.

(a) Show that $r(M \otimes_A N) = r(M)r(N)$.

(b) Let $I \subset A$ be an ideal with $r(I) > 1$. Show that $r(I^2) < r(I)^2$.

(c) Conclude that if every ideal in A is a flat module, then A is a PID (or a field).

5. Flat deformations.

(a) Suppose that k is a field and $k[\epsilon]$ is the ring of dual numbers $k[\epsilon] = k[x]/(x^2)$ and $\epsilon = \bar{x}$. Show that for any k-algebra A there is a flat $k[\epsilon]$-algebra B such that A is isomorphic to $B/\epsilon B$.

(b) Suppose that $k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and $A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^p, x_2^p, x_3^p, x_4^p, x_5^p, x_6^p)$. Show that there exists a flat $\mathbb{Z}/p^2\mathbb{Z}$-algebra B such that B/pB is isomorphic to A. (So here p plays the role of ϵ.)

(c) Now let $p = 2$ and consider the same question for $k = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and $A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2 + x_3x_4 + x_5x_6)$. However, in this case show that there does *not* exist a flat $\mathbb{Z}/4\mathbb{Z}$-algebra B such that $B/2B$ is isomorphic to A. (Find the trick! The same example works in arbitrary characteristic $p > 0$, except that the computation is more difficult.)