Let $\phi : A \to B$ be a homomorphism of rings. We say that the going-up theorem holds for $\phi$ if the following condition is satisfied:

(GU) for any $p, p' \in \text{Spec}(A)$ such that $p \subset p'$, and for any $P \in \text{Spec}(B)$ lying over $p$, there exists $P' \in \text{Spec}(B)$ lying over $p'$ such that $P \subset P'$.

Similarly, we say that the going-down theorem holds for $\phi$ if the following condition is satisfied:

(GD) for any $p, p' \in \text{Spec}(A)$ such that $p \subset p'$, and for any $P' \in \text{Spec}(B)$ lying over $p'$, there exists $P \in \text{Spec}(B)$ lying over $p$ such that $P \subset P'$.

1. In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)
   (a) $k$ is a field, $A = k, B = k[x]$.
   (b) $k$ is a field, $A = k[x], B = k[x, y]$.
   (c) $A = \mathbb{Z}, B = \mathbb{Z}[1/11]$.
   (d) $k$ is an algebraically closed field, $A = k[x, y], B = k[x, y, z]/(x^2 - y, z^2 - x)$.
   (e) $A = \mathbb{Z}, B = \mathbb{Z}[i, 1/(2 + i)]$.
   (f) $A = \mathbb{Z}, B = \mathbb{Z}[i, 1/(14 + 7i)]$.
   (g) $k$ is an algebraically closed field, $A = k[x], B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

2. Let $k$ be an algebraically closed field. Compute the image in $\text{Spec}(k[x, y])$ of the following maps:
   (a) $\text{Spec}(k[x, yx^{-1}]) \to \text{Spec}(k[x, y]),$ where $k[x, y] \subset k[x, yx^{-1}] \subset k[x, y, x^{-1}]$. (Hint: To avoid confusion, give the element $yx^{-1}$ another name.)
   (b) $\text{Spec}(k[x, y, a, b]/(ax - by - 1)) \to \text{Spec}(k[x, y])$.
   (c) $\text{Spec}(k[t, 1/(t - 1)]) \to \text{Spec}(k[x, y]),$ induced by $x \mapsto t^2$, and $y \mapsto t^3$.
   (d) $k = \mathbb{C}$ (complex numbers), $\text{Spec}(k[s, t]/(s^3 + t^3 - 1)) \to \text{Spec}(k[x, y]),$ where $x \mapsto s^2, y \mapsto t^2$.

Remark. Finding the image as above usually is done by using elimination theory.

3. Let $k$ be a field. Show that the following pairs of $k$-algebras are not isomorphic:
   (a) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
   (b) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
   (c) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

Remark. Of course the idea of this exercise is to find a simple argument in each case rather than applying a “big” theorem. Nonetheless it is good to be guided by general principles.