Commutative Algebra

Excercises 4

1. *A silly argument using the complex numbers!* Let $\mathbb{C}$ be the complex number field. Let V be a vector space over $\mathbb{C}$. The spectrum of a linear operator $T : V \to V$ is the set of complex numbers $\lambda \in \mathbb{C}$ such that the operator $T - \lambda \mathrm{id}_V$ is not invertible.

(a) Show that $\mathbb{C}(X) = f.f.(\mathbb{C}[X])$ has uncountable dimension over $\mathbb{C}$.
(b) Show that any linear operator on V has a nonempty spectrum if the dimension of V is finite or countable.
(c) Show that if a finitely generated $\mathbb{C}$-algebra R is a field, then the map $\mathbb{C} \to R$ is an isomorphism.
(d) Show that any maximal ideal $\mathfrak{m}$ of $\mathbb{C}[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in \mathbb{C}$.

Remark. Let k be a field. Then

(†) for every integer $n \in \mathbb{N}$ and every maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k.

This will be shown later in the course. Of course (please check this) it implies a similar statement for maximal ideals of finitely generated k-algebras. The exercise above proves (†) in the case $k = \mathbb{C}$.

2. Let k be a field. Please use (†) in (b) below.

(a) Let R be a k-algebra. Suppose that $\dim_k R < \infty$ and that R is a domain. Show that R is a field.
(b) Suppose that R is a finitely generated k-algebra, and $f \in R$ not nilpotent. Show that there exists a maximal ideal $\mathfrak{m} \subset R$ with $f \notin \mathfrak{m}$.
(c) Show by an example that this statement fails when R is not of finite type over a field.
(d) Show that any radical ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it.

Remark. This is the Hilbert Nullstellensatz. Namely it says that the closed subsets of $\operatorname{Spec} k[x_1, \ldots, x_n]$ (which correspond to radical ideals by a previous exercise) are determined by the closed points contained in them.

3. Let $A = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Let I be the ideal of A generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Find the irreducible components of the closed subset $V(I)$ of $\operatorname{Spec} A$. (I mean describe them and give equations for each of them. You do not have to prove that the equations you write down define prime ideals.) Hints:

(1) You may use the Hilbert Nullstellensatz, and it suffices to find irreducible locally closed subsets which cover the set of closed points of $V(I)$.
(2) There are two easy components.
(3) An image of an irreducible set under a continuous map is irreducible.