Commutative Algebra

Excercises 6

1. Suppose that k is a field having a primitive nth root of unity ζ. This means that $\zeta^n = 1$, but $\zeta^m \neq 1$ for $0 < m < n$.

(a) Show that the characteristic of k is prime to n.

(b) Suppose that $a \in k$ is an element of k which is not an dth power in k for any divisor d of n, $in \geq d > 1$. Show that $k[x]/(x^n - a)$ is a field. (Hint: Consider a splitting field for $x^n - a$ and use Galois theory.)

2. Let $\nu : k[x] \setminus \{0\} \to \mathbb{Z}$ be a map with the following properties: $\nu(fg) = \nu(f) + \nu(g)$ whenever f, g not zero, and $\nu(f+g) \geq min(\nu(f), \nu(g))$ whenever f, g, $f+g$ are not zero, and $\nu(c) = 0$ for all $c \in k^*$.

(a) Show that if f, g, and $f+g$ are nonzero and $\nu(f) \neq \nu(g)$ then we have equality $\nu(f+g) = min(\nu(f), \nu(g))$.

(b) Show that if $f = \sum a_i x^i$, $f \neq 0$, then $\nu(f) \geq min(\{i\nu(x)\}_{a_i \neq 0})$. When does equality hold?

(c) Show that if ν attains a negative value then $\nu(f) = -n \deg(f)$ for some $n \in \mathbb{N}$.

(d) Suppose $\nu(x) \geq 0$. Show that $\{f \mid f = 0,\ or\ \nu(f) > 0\}$ is a prime ideal of $k[x]$.

(e) Describe all possible ν.

3. Let $(A, \mathfrak{m}, k)$ be a local ring and let $k \subset k'$ be a finite field extension. Show there exists a flat, local map of local rings $A \to B$ such that $\mathfrak{m}_B = \mathfrak{m}B$ and $B/\mathfrak{m}B$ is isomorphic to k' as k-algebra. (Hint: first do the case where $k \subset k'$ is generated by a single element.)

Remark. The same result holds for arbitrary field extensions $k \subset K$.

4. Let R be a ring and let M be a finitely presented R module. Recall this means that there is an exact sequence

$$R^{\oplus r} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

This is called a presentation of M. Note that the map $R^{\oplus n} \to M$ is given by a sequence of elements $x_1, \ldots, x_n$ of M. The elements x_i are generators of M. The map $R^{\oplus r} \to R^{\oplus n}$ is given by a $n \times r$ matrix A with coefficients in R. The columns of A are said to be the relations. Any vector $(r_i) \in R^{\oplus r}$ such that $\sum r_i x_i = 0$ is a linear combination of the columns of A.

Of course any module has a lot of different presentations. We define $Pres(M)$ to be the collection of matrices you can obtain in this way (meaning all matrices A of any size $n \times r$ that occur in some presentation of M).

(a) Show that if $A \in Pres(M)$ has size $n \times r$ then the matrix $\tilde{A}$ of size $n \times (r+1)$ obtained from A by adding a column of zeros occurs in $Pres(M)$. (Hint: this corresponds to adding a trivial relation.)

(b) Show that if $A \in Pres(M)$, then any $\tilde{A} \in Pres(M)$, where $\tilde{A}$ is obtained from A by replacing ith column vector A_i by $A_i + \sum_{j \neq i} r_j A_j$ for any $r_j \in R$. (Hint: This corresponds to replacing a relation by itself plus a linear combination of other relations.)

(c) Show that if $A \in Pres(M)$ has size $n \times r$ then the matrix $\tilde{A}$ of size $(n+1) \times (r+1)$ obtained from A by setting

$$\begin{aligned} \tilde{a}_{ij} &= a_{ij},\ i < n+1, j < r+1, \\ \tilde{a}_{n+1j} &= 0,\ j < r+1, \\ \tilde{a}_{ir+1} &= 0,\ i < n+1, \\ \tilde{a}_{n+1r+1} &= 1 \end{aligned}$$

occurs in $Pres(M)$. (Hint: This corresponds to adding $x_{n+1} = 0$ and the trivial relation $x_{n+1} = 0$.)

(d) Show that if $A \in Pres(M)$, then $\tilde{A} \in Pres(M)$ where $\tilde{A}$ is obtained from A by replacing the jth row by a sum consisting of itself and a linear combination of other rows (with coefficients in R). (Hint: This corresponds to replacing x_j by $x_j + \sum_{i \neq j} r_i x_i$ and adjusting the relations accordingly.)

We say that matrices A and A' with coefficients in R are obtained from each other by a sequence of elementary moves if there is a sequence of matrices $A = A_0, A_1, A_2, \ldots, A_n = A'$ such that for each $0 \leq \ell < n$ the pair $(A_\ell, A_{\ell+1})$ is the pair $(A, \tilde{A})$ or $(\tilde{A}, A)$ for one of the operations on matrices described in (a)-(d) above.

(e) Show that any two matrices in $Pres(M)$ are obtained from each other by a sequence of elementary moves. (Hint: First show this holds if A, A' in $Pres(M)$ are matrices of relations among the same set of generators.)

(f) Let k be an integer. Suppose that A, A' are obtained from each other by a sequence of elementary moves. Say A has size $n \times r$ and A' has size $n' \times r'$. Show that the ideal generated by the $(n-k) \times (n-k)$ minors of A agrees with the ideal generated by the $(n'-k) \times (n'-k)$ minors of A'. [[Convention: If $k \geq n$ then we say the ideal generated by the $(n-k) \times (n-k)$-minors is R. In other words, the determinant of a matrix of size 0×0, -1×-1, etc is defined to be 1.]]

This defines the kth fitting ideal of M. Notation $Fit_k(M)$.

(g) Show that $Fit_0(M) \subset Fit_1(M) \subset Fit_2(M) \subset \ldots$. (Hint: Use that a determinant can be computed by expanding along a column.)

(h) Show that $M = (0)$ if $Fit_0(M) = R$.

(i) Show that M if $Fit_0(M) = (0)$ and $Fit_1(M) = R$, then M is locally free of rank 1. (This is slightly tricky.)