Commutative Algebra

Excercises 7

A *numerical polynomial* is a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(n) \in \mathbb{Z}$ for every integer n. A *graded module* M over a ring A is an A-module M endowed with a direct sum decomposition $\bigoplus_{n\in\mathbb{Z}} M_n$ into A-submodules. We will say that M is *locally finite* if all of the M_n are finite A-modules. Suppose that A is a Noetherian ring and that φ is a *Euler-Poincaré function* on finite A-modules. This means that for every finitely generated A-module M we are given an integer $\varphi(M) \in \mathbb{Z}$ and for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have $\varphi(M) = \varphi(M') + \varphi(M')$. The *Hilbert function* of a locally finite graded module M (with respect to φ) is the function $\chi_\varphi(M, n) = \varphi(M_n)$. We say that M has a *Hilbert polynomial* if there is some numerical polynomial P_φ such that $\chi_\varphi(M, n) = P_\varphi(n)$ for all sufficiently large integers n.

A *graded A-algebra* is a graded A-module $B = \bigoplus B_n$ together with an A-bilinear map

$$B \times B \longrightarrow B,\ (b, b') \longmapsto bb'$$

that turns B into an A-algebra so that $B_n \cdot B_m \subset B_{n+m}$. Finally, a *graded module M over a graded A-algebra B* is given by a graded A-module M together with a (compatible) B-module structure such that $B_n \cdot M_d \subset M_{n+d}$. Now you can define *homomorphisms of graded modules/rings*, *graded submodules*, *graded ideals*, *exact sequences of graded modules*, etc, etc.

1. Let $A = k$ a field. What are all possible Euler-Poincaré functions on finite A-modules in this case?

2. Let $A = \mathbb{Z}$. What are all possible Euler-Poincaré functions on finite A-modules in this case?

3. Let $A = k[x, y]/(xy)$ with k algebraically closed. What are all possible Euler-Poincaré functions on finite A-modules in this case?

4. Suppose that A is Noetherian. Show that the kernel of a map of locally finite graded A-modules is locally finite.

5. Let k be a field and let $A = k$ and $B = k[x, y]$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

6. Let k be a field and let $A = k$ and $B = k[x, y]/(x^2, xy)$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

7. Let k be a field and let $A = k$. Let $\varphi(M) = \dim_k(M)$. Fix $d \in \mathbb{N}$. Consider the graded A-algebra $B = k[x, y, z]/(x^d + y^d + z^d)$, where x, y, z each have degree 1. Compute the Hilbert function of B. Is there a Hilbert polynomial in this case?

To be continued.