Let $R$ be a graded ring. A homogeneous ideal is simply an ideal $I \subset R$ which is also a graded submodule of $R$. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of $f$ into homogenous pieces in $R$ then $f_i \in I$ for each $i$. We define $\text{Proj}(R)$ to be the set of homogenous, prime ideals $p$ of $R$ such that $R_p \not\subset p$. Note that $\text{Proj}(R)$ is a subset of $\text{Spec}(R)$ and hence has a natural induced topology.

Let $R = \bigoplus_{d \geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R(f)$ to be the subring of $R$ consisting of elements of the form $r/f^n$ with $r$ homogenous and $\deg(r) = nd$. Furthermore, we define $D(f) = \{p \in \text{Proj}(R)|f \not\in p\}$.

Finally, for a homogenous ideal $I \subset R$ we define $V(I) = V(I) \cap \text{Proj}(R)$.

1. Topology on $\text{Proj}(R)$. With notations as above:
   (a) Show that $D(f)$ is open in $\text{Proj}(R)$, show that $D(ff') = D(f) \cap D(f')$.
   (b) Let $g = g_0 + \ldots + g_m$ be an element of $R$ with $g_i \in R_i$. Express $D(g) \cap \text{Proj}(R)$ in terms of $D(g_i)$, $i \geq 1$ and $D(g_0) \cap \text{Proj}(R)$. No proof necessary.
   (c) Let $g \in R_0$ be a homogenous element of degree 0. Express $D(g) \cap \text{Proj}(R)$ in terms of $D(f_{\alpha})$ for a suitable family $f_{\alpha} \in R$ of homogenous elements of positive degree.
   (d) Show that the collection $\{D(f)|f\}$ of opens forms a basis for the topology of $\text{Proj}(R)$.
   (e) Show that there is a canonical bijection $D(f) \to \text{Spec}(R(f))$.
   (f) Show that the map from (e) is a homeomorphism.
   (g) Give an example of an $R$ such that $\text{Proj}(R)$ is not quasi-compact. No proof necessary.
   (i) Show that any closed subset $T \subset \text{Proj}(R)$ is of the form $V(I)$ for some homogenous ideal $I \subset R$.

Remark. There is a continuous map $\text{Proj}(R) \to \text{Spec}(R_0)$.

2. If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\text{Proj}(R) \to \text{Spec}(A)$ is a bijection and in fact a homeomorphism.

3. Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \text{Proj}(R) \to \text{Spec}(A)$. Set $U = \text{Spec}(A) - V(I)$. Show that

$$b : b^{-1}(U) \to U$$

is a homeomorphism.

Thus we may think of $U$ as an open subset of $\text{Proj}(R)$. Let $Z \subset \text{Spec}(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \not\in V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \not\in U$, in other words $Z \cap U \neq \emptyset$. We define the strict transform
4. Blowing up: Part II. Let $A = k[x, y]$ where $k$ is a field, and let $I = (x, y)$. Let $R$ be the blow up algebra for $A$ and $I$.
   (a) Show that the strict transforms of $Z_1 = V\{x\}$ and $Z_2 = V\{y\}$ are disjoint.
   (b) Show that the strict transforms of $Z_1 = V\{x\}$ and $Z_2 = V\{x - y^2\}$ are not disjoint.
   (c) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V\{x\}$ and $Z_2 = V\{x - y^2\}$ are disjoint.

5. Let $R$ be a graded ring.
   (a) Show that Proj($R$) is empty if $R_n = (0)$ for all $n >> 0$.
   (b) Show that Proj($R$) is an irreducible topological space if $R$ is a domain and $R_+$ is not zero. (Recall that the empty topological space is not irreducible.)

   Let $Z = V(p)$ for some prime ideal $p$. Let $\bar{A} = A/p$ and let $\bar{I}$ be the image of $I$ in $\bar{A}$.
   (a) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
   (b) Show that the ring map above induces a bijective map from Proj($\bar{R}$) onto the strict transform $Z'$ of $Z$. (This is not so easy. Hint: Use 5(b) above.)
   (c) Conclude that the strict transform $Z' = V_+ (P)$ where $P \subset R$ is the homogenous ideal defined by $P_+ = I^d \cap p$.
   (d) Suppose that $Z_1 = V(p)$ and $Z_2 = V(q)$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = p + q$ separates the strict transforms of $Z_1$ and $Z_2$, i.e., $Z'_1 \cap Z'_2 = \emptyset$. (Hint: Consider the homogenous ideal $P$ and $Q$ from part (c) and consider $V(P + Q)$.)