A Noetherian local ring \( A \) is said to be **Cohen-Macaulay** of dimension \( d \) if it has dimension \( d \) and there exists a system of parameters \( x_1, \ldots, x_d \) for \( A \) such that \( x_i \) is a nonzero divisor in \( A/(x_1, \ldots, x_{i-1}) \) for \( i = 1, \ldots, d \).

1. **Cohen-Macauley rings of dimension 1. Part I: Theory.**
   (a) Let \((A, \mathfrak{m})\) be a local Noetherian with \( \dim A = 1 \). Show that if \( x \in \mathfrak{m} \) is not a zero divisor then
   
   (1) \( \dim A/xA = 0 \), in other words \( A/xA \) is Artinian, in other words \( \{x\} \) is a system of parameters for \( A \).
   (2) \( A \) is has no embedded prime.

   (b) Conversely, let \((A, \mathfrak{m})\) be a local Noetherian ring of dimension 1. Show that if \( A \) has no embedded prime then there exists a nonzero divisor in \( \mathfrak{m} \).

2. **Cohen-Macauley rings of dimension 1. Part II: Examples.**
   b. Let \( A \) be the local ring at \((x, y)\) of \( k[x, y]/(x^2, xy) \).
      
      (1) Show that \( A \) has dimension 1.
      (2) Prove that every element of \( \mathfrak{m} \subset A \) is a zero divisor.
      (3) Find \( z \in \mathfrak{m} \) such that \( \dim A/zA = 0 \) (no proof required).

   c. Let \( A \) be the local ring at \((x, y)\) of \( k[x, y]/(x^2) \). Find a nonzero divisor in \( \mathfrak{m} \) (no proof required).

3. **Local rings of embedding dimension 1.** Suppose that \((A, \mathfrak{m}, k)\) is a Noetherian local ring of embedding dimension 1, i.e.,

   \[ \dim_k \mathfrak{m}/\mathfrak{m}^2 = 1. \]

   Show that the function \( f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1} \) is either constant with value 1, or its values are 

   \[ 1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots \]

4. **Regular local rings of dimension 1.** Suppose that \((A, \mathfrak{m}, k)\) is a regular Noetherian local ring of dimension 1. Recall that this means that \( A \) has dimension 1 and embedding dimension 1, i.e.,

   \[ \dim_k \mathfrak{m}/\mathfrak{m}^2 = 1. \]

   Let \( x \in \mathfrak{m} \) be any element whose class in \( \mathfrak{m}/\mathfrak{m}^2 \) is not zero.
   (a) Show that for every element \( y \) of \( \mathfrak{m} \) there exists an integer \( n \) such that \( y \) can be written as \( y = ux^n \) with \( u \in A^* \) a unit.
   (b) Show that \( x \) is a nonzero divisor in \( A \).
   (c) Conclude that \( A \) is a domain.

5. **Let \((A, \mathfrak{m}, k)\) be a Noetherian local ring with associated graded \( Gr_{\mathfrak{m}}(A) \).**
   (a) Suppose that \( x \in \mathfrak{m}^d \) maps to a nonzero divisor \( \bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1} \) in degree \( d \) of \( Gr_{\mathfrak{m}}(A) \).
   Show that \( x \) is a nonzero divisor.
(b) Suppose the depth of $A$ is at least 1. Namely, suppose that there exists a nonzero divisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree $d$ of $Gr_{\mathfrak{m}}(A)$ which is a nonzero divisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map $\mathfrak{m}^n/\mathfrak{m}^{n+1} \xrightarrow{x} \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$ is injective. Then show that $x$ is a nonzero divisor.

6. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of $A$ is 2, i.e., assume that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$.

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogenous ideal $I$.

(a) Show that there exists a homogenous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.

(b) Show that $f(n) \leq n + 1$.

(c) Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.

(d) Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.

(e) Show that $f(n) = \deg(F)$ for all $n >> 0$.

7. Cohen-Macauley rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macauley of dimension 1. Assume also that the embedding dimension of $A$ is 2, i.e., assume that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$.

Notations: $f$, $F$, $x, y \in \mathfrak{m}$, $I$ as in Ex. 6 above. Please use any results from the problems above.

(a) Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha \bar{x} + \beta \bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with $f$.

(1) Show that $z$ is a nonzero divisor on $A$.

(2) Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d} \mathfrak{m}^{d-1}$ for all sufficiently large $n$. (Hint: First show $z^{n+1-d} \mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)

(b) What condition on $k$ guarantees the existence of such a $z$? (No proof required; it’s too easy.)

Now we are going to assume there exists a $z$ as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).

(c) Now show that $\mathfrak{m}^\ell = z^{\ell-d+1} \mathfrak{m}^{d-1}$ for all $\ell \geq d$.

(d) Conclude that $I = (F)$.

(e) Conclude that the function $f$ has values

$$2, 3, 4, \ldots, d - 1, d, d, d, d, d, d, d, d, d, \ldots$$

This suggests that a local Noetherian Cohen-Macauley ring of dimension 1 and embedding dimension 2 is of the form $B/FB$, where $B$ is a 2-dimensional regular local ring. This is more or less true (under suitable “niceness” properties of the ring).