Throughout $k$ is a field. For any finite type $k$-algebra $A$ we consider the function

$$\delta : \text{Spec}(A) \to \mathbb{Z}, \quad p \mapsto \delta(p) = \text{trdeg}_k(\kappa(p))$$

**Lemma 0.1.** Let $A \subset B$ be an integral extension of domains. If $J \subset B$ is a nonzero ideal, then $A \cap J$ is nonzero.

*Proof.* Let $b \in J$ be nonzero. Let $P = t^d + a_1t^{d-1} + \ldots + a_d \in A[t]$ be a monic polynomial of minimal degree with $P(b) = 0$. Then $a_d$ is nonzero, since otherwise we may replace $P$ by $P/t$ (here we use that $B$ is a domain and $b$ is nonzero) and lower the degree of $P$. Then $a_d = -b^d - a_1b^{d-1} - \ldots - a_{d-1}b \in J \cap A$. \hfill $\square$

**Lemma 0.2.** Let $A \subset B$ be an extension of domains and denote $K \subset L$ their fraction fields. Let $b \in B$ be integral over $A$. If $A$ is normal, then the minimal polynomial $P \in K[t]$ of $b$ viewed as an element of $L$ has coefficients in $A$.

*Proof.* Let $Q \in A[t]$ be a monic polynomial such that $Q(b) = 0$. Since $P$ is the minimal polynomial of $b$ we see that $P$ is monic and $P$ divides $Q$ in $K[t]$. Choose a splitting field $M/L$ of $Q$. Write

$$Q = (t - \beta_1) \ldots (t - \beta_m)$$

with $\beta_1, \ldots, \beta_m$ the roots of $Q$ in $M$. Clearly, $\beta_1, \ldots, \beta_m$ are integral over $A$. Since $P|Q$ and $P$ is monic, we see that after a permutation we can find $n \leq m$ such that

$$P = (t - \beta_1) \ldots (t - \beta_n)$$

Thus the coefficients are polynomials in the $\beta_j$’s and in particular they are integral over $A$. Since they are also in $K$ and $A$ is normal, we see that they are in $A$. \hfill $\square$

**Lemma 0.3.** Let $A$ be a finite type $k$-algebra. For $g \in A$ there is a finite type $k$-algebra $A_g = A[t]/(gt - 1)$ with the following property: the induced map $\text{Spec}(A_g) \to \text{Spec}(A)$ induces a homeomorphism onto $D(g) \subset \text{Spec}(A)$ and identifies residue fields.

*Proof.* Omitted. \hfill $\square$

**Lemma 0.4.** Let $A$ be of finite type over a field $k$. Let $q \subset p \subset A$ be distinct primes. Then $\delta(q) > \delta(p)$.

*Proof.* Set $A' = A/q$, $q' = (0)$, and $p' = p/q$. Then $\kappa(p') = \kappa(p)$ and $\kappa(q') = \kappa(q)$; please prove this yourselves. Thus we may assume $A$ is a domain and $q = (0)$. In other words, we have to show: if $A$ is a domain and $p \subset A$ is a nonzero prime, then the transcendence degree of $\kappa(p)$ over $k$ is less than the transcendence degree of $f.f.(A)$.

Choose $P = k[x_1, \ldots, x_r] \subset A$ finite (Noether normalization, see previous lecture). Then

$$k(x_1, \ldots, x_r) \subset f.f.(A)$$

The existence of $Q$ guarantees that $b$ viewed as an element of $L$ is indeed algebraic over $K$ and hence the statement of the lemma makes sense.
is a finite extension, hence the transcendence degree of \( f.f.(A) \) is \( r \). By Lemma 0.1 there is a nonzero element \( g \in k[x_1, \ldots, x_r] \) contained in \( p \). Thus the images \( x_1, \ldots, x_r \) of the elements \( x_1, \ldots, x_r \) in \( \kappa(p) = f.f.(A/p) \) aren’t algebraically independent. Now \( \kappa(p) \) is algebraic over the subfield generated by \( k \) and \( x_1, \ldots, x_r \); please prove this yourselves. We conclude that the transcendence degree of \( \kappa(p) \) over \( k \) is strictly less than \( r \).

\[ \square \]

**Lemma 0.5.** Let \( A \) be of finite type over a field \( k \). Let \( q \subset p \subset A \) be distinct prime ideals with no prime ideal strictly in between. Then \( \delta(p) + 1 = \delta(q) \).

*Proof.* After replacing \( A \) by \( A/q \) as in the proof of Lemma 0.4 we may assume \( q = (0) \) and hence \( p \subset A \) is a prime ideal minimal with the property of not being zero. Our goal is to show that \( \text{trdeg}_k \kappa(p) + 1 = \text{trdeg}_k f.f.(A/p) \).

Pick nonzero \( f \) in \( p \). Then \( p \) is minimal over \( (f) \), i.e., \( p \) defines a generic point of \( V(f) \). By a previous result \( V(f) \) has a finite number of generic points besides \( p \). By prime avoidance can pick \( g \) in those primes but not in \( p \). After replacing \( A \) by \( A_g \) we get \( p = \sqrt{(f)} \). Some details omitted; see Lemma 0.3 to see why it is permissible to do this replacement.

Choose \( P = k[x_1, \ldots, x_r] \subset A \) finite (Noether normalization). Then \( r = \text{trdeg}_k f.f.(A) \), see proof of Lemma 0.4. For all elements of \( A \) the minimal polynomial has coefficients in \( P \) by Lemma 0.2 and the normality of the polynomial algebra. In particular get \( Nm : A \to P \) because the norm of an element is a power of the last coefficient of the minimal polynomial (see previous lecture).

Let \( q = P \cap p \). Then \( g \in q \) implies \( g^n = af \) for some \( n > 0 \) and \( a \in A \). Let \( d \) be the degree of the extension \( f.f.(A)/f.f.(P) \). Then

\[
g^{nd} = Nm(g^n) = Nm(g^n) = Nm(a)Nm(f)
\]

by properties of the norm (see previous lecture).

We have \( Nm(f) \in q \): the last coefficient of its minimal polynomial is in \( p \) and in \( P \) and \( Nm(f) \) is a power of it (compare with the proof of Lemma 0.1). Thus, we see that one of the irreducible factors, say \( g \), of \( Nm(f) \) is in \( q \). Applying the displayed equation above we see that \( Nm(f) \) is a power of the irreducible \( g \) up to a unit.

We claim \( q = (g) \). Namely, if \( h \in q \), then applying the displayed equation for \( h \) we see that \( g \) divides a power of \( h \), hence \( h \in (g) \).

The transcendence degree of \( \kappa(q) \) over \( k \) is \( r - 1 \) (see previous lecture). The extension \( \kappa(q) \subset \kappa(p) \) is finite because \( P \subset A \) is finite; please prove this yourselves. Thus the transcendence degree of \( \kappa(p) \) over \( k \) is \( r - 1 \) as well. \[ \square \]

**Lemma 0.6** (Hilbert Nullstellensatz). Let \( A \) be finite type over a field \( k \). Let \( p \subset A \) be a prime. Then the following are equivalent

1. \( p \) is a maximal ideal,
2. \( \text{trdeg}_k \kappa(p) = 0 \),
3. \( \kappa(p) \) is finite over \( k \).

*Proof.* After replacing \( A \) by \( A/p \) we see that we have to show the following: given a domain \( A \) of finite type over \( k \) the following are equivalent

1. \( A \) is a field,
2. \( \text{trdeg}_k f.f.(A) = 0 \),
Theorem 0.7. Let \( A \) be a finite type \( k \)-algebra and set \( X = \text{Spec}(A) \). The function 
\[
\delta : \text{Spec}(A) \rightarrow \mathbb{Z}, \quad p \mapsto \delta(p) = \text{trdeg}_k(\kappa(p))
\]
is a dimension function and we have \( \delta(p) = 0 \) if and only if \( p \) is a closed point. 

Proof. Combine Lemmas 0.4, 0.5, and 0.6. 

There are many things you can conclude from this. Let us give three examples.

Lemma 0.8. Let \( A \) be a finite type \( k \)-algebra and set \( X = \text{Spec}(A) \).

1. If \( x \in X \), then there is a specialization \( x \leadsto y \) with \( y \) closed in \( X \).
2. We have \( \dim(X) = \max\{\delta(x) \mid x \in X\} \).
3. For \( x \in X \) with \( \delta(x) > 0 \) the set of closed points of \( Z = \{x\} \) is infinite.

Proof. The proof of (1) is formal from the theorem. Namely, if \( \delta(x) = 0 \), then we take \( y = x \) (this is forced). If \( \delta(x) > 0 \) we see that \( x \) is not a closed point of \( X \). Hence we can find \( x \leadsto x' \) in \( X \) with \( \delta(x) > \delta(x') \). If \( \delta(x') > 0 \), then we can do it again. Thus we can continue
\[
x \leadsto x' \leadsto x'' \leadsto \ldots \leadsto x^{(e)}
\]
until we hit a final point \( x^{(e)} \in X \) with \( \delta(x^{(e)}) = 0 \). Set \( y = x^{(e)} \). This is a closed point by the theorem. Since specialization is transitive we see that \( x \leadsto y \).

Proof of (2). Since \( X \) is a sober\(^2\) topological space the dimension of \( X \) is equal to the supremum of the lengths of chains of nontrivial specializations
\[
x_n \leadsto x_{n-1} \leadsto \ldots \leadsto x_0
\]
in \( X \). By the properties of a dimension function, we can assume \( \delta(x_i) = \delta(x_{i-1}) + 1 \). Since every prime in a ring contains a minimal prime and is contained in a maximal ideal, we may assume that \( x_n \) is a generic point of an irreducible component of \( X \) and \( x_0 \) a closed point. Then \( n = \delta(x_n) \). Finally, there are only finitely many irreducible components of \( X \) hence the supremum is attained.

To prove (3) we use that \( X \) has a basis for its topology consisting of opens \( U \) which also have the property with respect to \( \delta_U \); namely the principal opens \( D(y) = \text{Spec}(A_y) \), see Lemma 0.3. Suppose we have closed points \( x_1, \ldots, x_n \in Z \). Then we can choose an open \( U \subseteq X \) as above with \( x \in U \) and \( x_1 \notin U, \ldots, x_n \notin U \). By (1) applied to \( x \in U \) we can find \( x \leadsto y \) in \( U \) with \( y \) closed in \( U \), equivalent \( \delta(y) = 0 \). Then \( y \in X \) is closed because the function \( \delta \) is the same for \( y \) viewed as a point

\(^2\)This means that every closed irreducible subset \( Z \) is of the form \( \overline{\{\eta\}} \) for a unique generic point \( \eta \in Z \).
of $U$ or of $X$! Since $y \in Z$ is not equal to $x_1, \ldots, x_n$ it is a “new” closed point as desired.

**Example 0.9.** For example, part (3) of the last lemma says that $k[x_1, \ldots, x_n]$ has infinitely many maximal ideals if $n > 0$. This is obvious if $k$ is infinite, as you can take the ideals $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in k^n$. But it is true even if $k$ is a finite field. More interestingly perhaps, suppose $k = \mathbb{Q}$ and consider $A = \mathbb{Q}[x, y]/(x^2 + y^2 + 1)$. Then there are no $\mathbb{Q}$-rational points on $\text{Spec}(A)$, i.e., there are no maximal ideals $m \subset A$ with $\kappa(m) \cong \mathbb{Q}$. However, there are still infinitely many maximal ideals: you can take $m_n = (x - n, y^2 + n^2 + 1)$ for $n \in \mathbb{N}$.

**Example 0.10.** Suppose that we consider the ring

$$A = k[x, y, z, w]/(xy, xz)$$

Then we have two irreducible components corresponding to the prime ideals $(x)$ and $(y, z)$. The dimension of these irreducible components is 3 and 2. Thus the dimension of $A$ is 3.