Schemes

Exercises 3

Schemes – examples are important

Please also argue that your examples are as required. Feel free to quote results from Hartshorne or EGA.

1. Give an example of a morphism of *integral* schemes $f : X \to Y$ such that the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$, but f is not a closed immersion.

2. Give examples of graded rings S such that
(a) $\mathrm{Proj}(S)$ is affine and nonempty, and
(b) $\mathrm{Proj}(S)$ is integral, nonempty but not isomorphic to $\mathbb{P}^n_A$ for any $n \geq 0$, any ring A.

3. Give an example of a nonconstant morphism of schemes $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^5_{\mathbb{C}}$ over $\mathrm{Spec}(\mathbb{C})$.

4. Give an example of an isomorphism of schemes $\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Proj}(\mathbb{C}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))$.

5. Give an example of a morphism of schemes $f : X \to \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[T])$ such that the (scheme theoretic) fibre of f over $t \in \mathbb{A}^1_{\mathbb{C}}$ is (a) isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ when t is a closed point not equal to 0, and (b) not isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ when $t = 0$.

Remark. This can be done in many, many ways. Here are some additional restraints you can impose: Can you do it with fibre at $t = 0$ projective? Can you do it with special fibre irreducible and projective? Can you do it with special fibre integral and projective? Can you do it with fibre at $t = 0$ smooth and projective? What about similar questions when you replace $\mathbb{P}^1_{\mathbb{C}}$ with another variety over $\mathbb{C}$?