Schemes

Exercises 4

Schemes – examples are important

Please also argue that your examples are as required. Feel free to quote results from Hartshorne or EGA. It is fine to copy examples out of Hartshorne, and reference the explanation if it is in Hartshorne.

1. (Pretty hard. You can leave some of the verifications out if you like.) Give an example of a fibre product $X \times_S Y$ such that X and Y are affine but $X \times_S Y$ is not.

Remark/Hint. It turns out this cannot happen with S separated. Do you know why?

2. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over $\mathbb{Q}$ such that $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{Q}} V$ is not integral.

3. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over a field k such that $\operatorname{Spec} k' \times_{\operatorname{Spec} k} V$ is not reduced for some finite field extension $k \subset k'$.

Remark. If your scheme is affine then dimension is the same as the Krull dimension of the underlying ring. So you can use last semesters results to compute dimension.

4. Give an example of a surjective morphism $X \to \mathbb{P}^n_{\mathbb{C}}$ with X affine.

5. (For the number theorists.) Give an example of a closed subscheme

$$Z \subset \operatorname{Spec} \mathbb{Z}[x, \frac{1}{x(x-1)(2x-1)}]$$

such that the morphism $Z \to \operatorname{Spec} \mathbb{Z}$ is finite and surjective.

Remark. If you do not like number theory, you can try the variant where you look at

$$\operatorname{Spec} \mathbb{F}_p[t, x, \frac{1}{x(x-t)(tx-1)}] \longrightarrow \operatorname{Spec} \mathbb{F}_p[t]$$

and you try to find a closed subscheme of the top scheme which maps finite surjectively to the bottom one. (There is a theoretical reason for having a finite ground field here; allthough it may not be necessary in this particular case.)