Schemes

Exercises 6

Schemes – Examples again

Feel free to quote results from Hartshorne or EGA.

1. Čech cohomology. Here $k$ is a field.

   (a) Let $X$ be a scheme with an open covering $U : X = U_1 \cup U_2$, with $U_1 = \text{Spec } k[x]$, $U_2 = \text{Spec } k[y]$ with $U_1 \cap U_2 = \text{Spec } k[z, 1/z]$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We’ve seen in the lectures that such an $X$ exists; it is the affine line with zero doubled.) Compute $\check{H}^1(U, \mathcal{O})$; eg. give a basis for it as a $k$-vectorspace.

   (b) For each element in $\check{H}^1(U, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules

   $0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$

   such that the boundary $\delta(1) \in \check{H}^1(U, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. It is also OK to show abstractly such a thing has to exist.)

**Definition of delta.** Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of abelian sheaves on any topological space $X$. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $U : X = \bigcup_{i \in I} U_i$ such that for each $i$ there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of $\tau$ to $U_i$. Then consider the assignment

$$(i_0, i_1) \mapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$

This is clearly a 1-coboundary in the Čech complex $\check{C}^*(U, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $\check{C}^*(U, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

2. Algebra. (Silly and should be easy.)

   (a) Give an example of a ring $A$ and a nonsplit short exact sequence of $A$-modules

   $0 \to M_1 \to M_2 \to M_3 \to 0$.

   (b) Give an example of a nonsplit sequence of $A$-modules as above and a faithfully flat $A \to B$ such that

   $0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$.

   is split as a sequence of $B$-modules.

3. Maps of Proj. Let $R$ and $S$ be graded rings. So $R = \oplus_{d \geq 0} R_d$ and $R_a \cdot R_b \subset R_{a+b}$. Suppose we have a ring map

$$\varphi : R \to S$$

such that there exists an integer $e \geq 1$ such that $\varphi(R_d) \subset S_{de}$.

   (a) For which elements $p \in \text{Proj}(S)$ is there a well-defined corresponding point in $\text{Proj}(R)$? In other words, find a suitable open $U \subset \text{Proj}(S)$ such that $\varphi$ defines a continuous map $\text{Proj}(\varphi) : U \to \text{Proj}(R)$.

   (b) Give an example where $U \neq \text{Proj}(S)$.

   (c) Give an example where $U = \text{Proj}(S)$.

   (d) (Do not write this down.) Convince yourself that the continuous map $U \to \text{Proj}(R)$ comes canonically with a map on sheaves so that $\text{Proj}(\varphi)$ is a morphism of schemes:

$$\text{Proj}(S) \supset U \to \text{Proj}(R)$$
**Notation.** Let $R$ be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \text{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on $X$ such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (elements homogeneous of degree $n$ in $R_f = R[1/f]$). See Hartshorne, page 116+. Note that there are natural maps

$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

4. Pathologies in Proj. Give examples of $R$ as above such that

(a) $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.

(b) $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \rightarrow \mathcal{O}_X(2)$ is NOT an isomorphism.