Schemes

Exercises 6

Schemes – Divisors

Feel free to quote results from Hartshorne or EGA.

Definitions. Throughout, let X be a Noetherian, integral and separated scheme.

(a) A Weil divisor is a formal linear combination $\Sigma n_i[Z_i]$ of prime divisors Z_i with integer coefficients.

(b) A prime divisor is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. We will use the notation $\mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ when $\xi \in Z \subset X$ is as above. Note that $\mathcal{O}_{X,Z} \subset K(X)$ is a subring of the function field of X.

(c) The Weil divisor associated to a rational function $f \in K(X)^*$ is the sum $\Sigma v_Z(f)[Z]$. Here $v_Z(f)$ is defined as follows

(c1) If $f \in \mathcal{O}^*_{X,Z}$ then $v_Z(f) = 0$.

(c2) If $f \in \mathcal{O}_{X,Z}$ then

$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f)).$$

(c3) If $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{X,Z}$ then

$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

(d) An effective Cartier divisor on *any* scheme S is a closed subscheme $D \subset S$ such that every point $d \in D$ has an affine open neighbourhood $\operatorname{Spec} A = U \subset S$ in S so that $D \cap U = \operatorname{Spec} A/(f)$ with $f \in A$ a nonzero divisor.

(e) The Weil divisor $[D]$ associated to an effective Cartier divisor $D \subset X$ of our Noetherian integral separated scheme X is defined as the sum $\Sigma v_Z(D)[Z]$ where $v_Z(D)$ is defined as follows

(e1) If the generic point ξ of Z is not in D then $v_Z(D) = 0$.

(e1) If the generic point ξ of Z is in D then

$$v_Z(D) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f))$$

where $f \in \mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ is the nonzero divisor which defines D in an affine neighbourhood of ξ (as in definition **(d)** above).

(f) Let S be *any* scheme. The sheaf of total quotient rings $\mathcal{K}_S$ is the sheaf of $\mathcal{O}_S$-algebras which is the sheafification of the pre-sheaf $\mathcal{K}'$ defined as follows. For $U \subset S$ open we set $\mathcal{K}'(U) = S_U^{-1}\mathcal{O}_S(U)$ where $S_U \subset \mathcal{O}_S(U)$ is the multiplicative subset consisting of sections $f \in \mathcal{O}_S(U)$ such that the germ of f in $\mathcal{O}_{S,u}$ is a nonzero divisor for every $u \in U$. In particular the elements of S_U are all nonzero divisors. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{K}_S$, and we get a short exact sequence

$$0 \to \mathcal{O}^*_S \to \mathcal{K}^*_S \to \mathcal{K}^*_S/\mathcal{O}^*_S \to 0.$$

(g) A Cartier divisor on *any* scheme S is a global section of the quotient sheaf $\mathcal{K}^*_S/\mathcal{O}^*_S$.

(h) The Weil divisor associated to a Cartier divisor $\tau \in \Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X)$ over our Noetherian integral separated scheme X is the sum $\Sigma v_Z(\tau)[Z]$ where $v_Z(\tau)$ is defined as by the following recipe

(h1) If the germ of τ at the generic point ξ of Z is zero – in other words the image of τ in the stalk $(\mathcal{K}^*/\mathcal{O}^*)_\xi$ is "zero" – then $v_Z(\tau) = 0$.

(h2) Find an affine open neighbourhood $\operatorname{Spec} A = U \subset X$ so that $\tau|_U$ is the image of a section $f \in \mathcal{K}(U)$ and moreover $f = a/b$ with $a, b \in A$. Then we set

$$v_Z(f) = \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \text{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

Remarks. (a) On a Noetherian integral separated scheme X the sheaf $\mathcal{K}_X$ is constant with value the function field $K(X)$.

(b) To make sense out of the definitions above one needs to show that

$$\text{length}_{\mathcal{O}}(\mathcal{O}/(ab)) = \text{length}_{\mathcal{O}}(\mathcal{O}/(a)) + \text{length}_{\mathcal{O}}(\mathcal{O}/(b))$$

for any pair (a, b) of nonzero elements of a Noetherian 1-dimensional local domain $\mathcal{O}$. This will be done in the lectures.

1. Describe how to assign a Cartier divisor to an effective Cartier divisor.

The following questions have some logical dependencies; if you point them out then you won't have to do all of them.

2. Give an example of a Weil divisor (on a Noetherian integral separated scheme) which is not the Weil divisor associated to a rational function.

3. Give an example of a Weil divisor (on a Noetherian integral separated scheme) which is not the Weil divisor associated to any effective Cartier divisor.

4. Give an example of a Weil divisor (on a Noetherian integral separated scheme) which is not the Weil divisor associated to any Cartier divisor.

5. Give an example of a Weil divisor D (on a Noetherian integral separated scheme) which is not the Weil divisor associated to any Cartier divisor but such that nD is the Weil divisor associate to a Cartier divisor for some $n > 1$.

6. Give an example of a Weil divisor D (on a Noetherian integral separated scheme) which is not the Weil divisor associated to any Cartier divisor and such that nD is NOT the Weil divisor associate to a Cartier divisor for any $n > 1$.

7. Give an example of a Cartier divisor which is not the difference of (the Cartier divisors associated to) two effective Cartier divisors.†

† I do not know how to do this one myself, but I think this happens.