SHEAVES ON THE SPECTRUM OF A RING

Throughout $A$ is a ring. We set $X = \text{Spec}(A)$. We denote $\mathcal{B}$ the set of principal opens of $X$ (AKA standard opens). In a formula

$$\mathcal{B} = \{ U \subset X \mid \exists f \in A, \ U = D(f) \}$$

**Sheaves on a basis:** see Tag 009H

Think of $\mathcal{B}$ as a category: the objects are the elements of $\mathcal{B}$ and the morphisms are the inclusions. A presheaf $\mathcal{F}$ on $\mathcal{B}$ is a contravariant functor from $\mathcal{B}$ to the category of sets (or abelian groups, rings, etc). Similarly for presheaves of modules over a given presheaf of rings. We say $\mathcal{F}$ is a sheaf on $\mathcal{B}$ if and only if for every covering $U : U = U_1 \cup \ldots \cup U_n$ with $U, U_i \in \mathcal{B}$ we have that $\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightarrow \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1})$ is an equalizer diagram. Observe that this makes sense because $U, U' \in \mathcal{B} \Rightarrow U \cap U' \in \mathcal{B}$ as you can easily verify.

**Lemma 0.1.** The category of sheaves on $X$ and sheaves on $\mathcal{B}$ are equivalent via the functor which takes a sheaf on $X$ and restricts it to $\mathcal{B}$.

**Proof.** This is a copy of Tag 009O.

If $\mathcal{F}|_{\mathcal{B}}$ denotes the restriction of $\mathcal{F}$ on $X$ to $\mathcal{B}$ (as in the equivalence of Lemma 0.1), then we have for $x \in X$ the equality

$$\mathcal{F}_x = \text{colim}_{x \in U \in \mathcal{B}} \mathcal{F}(U) = \text{colim}_{x \in U \in \mathcal{B}} \mathcal{F}|_{\mathcal{B}}(U)$$

Hence we can directly compute the stalks in terms of the sheaf on $\mathcal{B}$.

**The structure sheaf:** see Tag 01HR

For every $U \in \mathcal{B}$ choose an element $f \in A$ such that $U = D(f)$. The we set

$$\mathcal{O}_X(U) = A_f$$

If $U = D(f) \supset V = D(g)$, then we can write $g^n = af$ for some $n > 0$ and $a \in A$ (small detail omitted) and we define the restriction mapping for $\mathcal{O}_X$ as the map of $A$-algebras

$$\mathcal{O}_X(U) = A_f \rightarrow A_g = \mathcal{O}_X(V)$$

sending $b/f^m$ to $ba^m/g^nm$. It is easy to see this is a presheaf of rings.

**The sheaf of modules $\widetilde{M}$ associate to an $A$-module $M$:** see Tag 01HR

Let $M$ be an $A$-module. For every $U \in \mathcal{B}$ choose an element $f \in A$ such that $U = D(f)$. The we set

$$\widetilde{M}(U) = M_f$$

---

1Since every element of $\mathcal{B}$ is quasi-compact we only need to consider finite coverings.
If \( U = D(f) \supset V = D(g) \), then we can write \( g^n = af \) for some \( n > 0 \) and \( a \in A \) (small detail omitted) and we define the restriction mapping for \( \tilde{M} \) as the map of \( A \)-modules

\[
\tilde{M}(U) = M_f \longrightarrow M_g = \tilde{M}(V)
\]

sending \( x/f^m \) to \( a x/g^{nm} \). It is easy to see this is a presheaf of modules over \( \mathcal{O}_X \). Also, observe that \( \mathcal{O}_X = \tilde{A} \), and hence if we prove \( \tilde{M} \) is a sheaf, then the same thing holds for \( \mathcal{O}_X \).

**Sheaf property:** To check the sheaf property consider a covering

\[ U : U = U_1 \cup \ldots \cup U_n \]

with \( U_i \in \mathcal{B} \). Write \( U = D(f) \) and \( U_i = D(f_i) \). To check the sheaf property it suffices to check

\[
0 \to M_f \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots
\]

is exact. The fact that \( U = \bigcup U_i \) implies that

\[
M_{f_{i_0} \ldots f_{i_p}} = (M_f)_{f_{i_0} \ldots f_{i_p}}
\]

and that \( f_1, \ldots, f_n \) generate the unit ideal in the ring \( A_f \). Hence the alternating Čech complex for \( U \) and \( \tilde{M} \) is the complex of Lemma 0.2 for the ring \( A_f \), the module \( M_f \), and the elements \( f_1/1, \ldots, f_n/1 \) of \( A_f \).

**Lemma 0.2.** Let \( A \) be a ring, let \( M \) be an \( A \)-module, let \( f_1, \ldots, f_n \in A \) generate the unit ideal. Then the complex

\[
0 \to M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots
\]

is exact.

**Proof.** Two steps: first if \( f_i \) is a unit in \( A \) for some \( i \), then one writes an explicit homotopy, next, one proves the lemma below to deduce the general case from this special case. \( \square \)

**Lemma 0.3.** Let \( A \) be a ring, let \( M_1 \to M_2 \to M_3 \) be a complex of \( A \)-modules, let \( f_1, \ldots, f_n \in A \) generate the unit ideal. Then \( M_1 \to M_2 \to M_3 \) is exact if and only if for each \( i \) the complex

\[
(M_i)_{f_i} \to (M_2)_{f_i} \to (M_3)_{f_i}
\]

is exact.

**Proof.** Using that localization is exact this reduces to the statement: if an \( A \)-module \( H \) satisfies \( H_{f_i} = 0 \) for \( i = 1, \ldots, n \), then \( H = 0 \). This is proved by considering an element \( x \in H \) and observing that the annihilator ideal of \( x \) contains \( f_i^{N_i} \) for some \( N_i > 0 \). Since \( f_1^{N_1}, \ldots, f_n^{N_n} \) is the unit ideal of \( A \), we conclude that \( x = 0 \). See Tag 00EN for more results of this nature. \( \square \)

**Proposition 0.4.** The higher cohomology groups of the structure sheaf \( \mathcal{O}_X \) and of the sheaves \( \tilde{M} \) vanish.

**Proof.** The discussion and lemmas above show that for any open covering \( U : U = U_1 \cup \ldots \cup U_n \) with \( U_i \in \mathcal{B} \) the higher Čech cohomology of \( \tilde{M} \) vanishes. Thus we may apply Tag 01EW. See also Tag 01XB. \( \square \)