A result of Gabber

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1 The result

Let X be a scheme endowed with an ample invertible sheaf $\mathcal{L}$. See EGA II, Definition 4.5.3. In particular, X is supposed quasi-compact and separated.

1.1 Theorem. *The cohomological Brauer group of X is equal to the Brauer group of X.*

The purpose of this note is to publish a proof of this result, which was prove by O. Gabber (private communication). The cohomological Brauer group $Br'(X)$ of X is the torsion in the étale cohomology group $H^2(X, \mathbb{G}_m)$. The Brauer group of X is the group of Azumaya algebras over X up to Morita equivalence. See [Hoobler] for more precise definitions.

Our proof, which is different from Gabber's proof, uses twisted sheaves. Indeed, a secondary goal of this paper is to show how using them some of the questions regarding the Brauer group are simplified. We do not claim any originality in defining α-twisted sheaves; these appear in work of Giraud, Caldararu, and Lieblich. Experts may skip Section 2 where we briefly explain what little we need about this notion.

2 Preliminaries

2.1 Let $(X, \mathcal{L})$ be a pair as in the Section 1. There exists a directed system of pairs $(X_i, \mathcal{L}_i)$ as in Section 1 such that $(X, \mathcal{L})$ is the inverse limit of the $(X_i, \mathcal{L}_i)$ and such that each X_i is of finite type over $\operatorname{Spec} \mathbb{Z}$. Furthermore, all the transition mappings in the system are affine. See [Thomason].

2.2 Let X, X_i be as in the previous paragraph. Then $Br'(X)$ is the direct limit of $Br'(X_i)$. This because étale cohomology commutes with limits. See ??.Similarly, $Br(X)$ is the limit of the groups $Br(X_i)$.

2.3 Let X be a scheme and let $\alpha \in H^2(X, \mathbb{G}_m)$. Suppose we can represent α by a Čech cocycle $\alpha_{ijk} \in \Gamma(U_i \times_X U_j \times_X U_k, \mathbb{G}_m)$ on some étale covering $\mathcal{U} : \{U_i \to X\}$ of X. An *α-twisted sheaf* is given by a system $(\mathcal{M}_i, \varphi_{ij})$, where each $\mathcal{M}_i$ is a quasi-coherent $\mathcal{O}_{U_i}$-module on U_i, and where $\varphi_{ij} : \mathcal{M}_i \otimes \mathcal{O}_{U_{ij}} \to \mathcal{M}_j \otimes \mathcal{O}_{U_{ij}}$ are isomorphisms such that

$$\varphi_{jk} \circ \varphi_{ij} = \alpha_{ijk} \varphi_{ik}$$

over U_{ijk}. Since we will be working with quasi-projective schemes all our cohomology classes will be represented by Čech cocycles, see [Artin]. In the general case one can define the category of α-twisted sheaves using cocycles with respect to hypercoverings; in 2.9 below we will suggest another definition and show that it is equivalent to the above in the case that there is a Čech cocycle.

2.4 We say that the α-twisted sheaf is coherent if the modules $\mathcal{M}_i$ are coherent. We say an α-twisted sheaf locally free if the modules $\mathcal{M}_i$ are locally free. Similarly, we can talk about finite or flat α-twisted sheaves.

2.5 Let $\alpha \in H^2(X, \mathbb{G}_m)$. Let $\mathcal{X} = \mathcal{X}_\alpha$ be the $\mathbb{G}_m$-gerb over X defined by the cohomology class α. This is an algebraic stack $\mathcal{X}$ endowed with a structure morphism $\mathcal{X} \to X$. Here is a quick way to define this gerb. Take an injective resolution $\mathbb{G}_m \to I^0 \to I^1 \to I^2 \to \ldots$ of the sheaf $\mathbb{G}_m$ on the big fppf site of X. The cohomology class α corresponds to a section $\tau \in \Gamma(X, I^2)$ with $\partial\tau = 0$. The stack $\mathcal{X}$ is a category whose objects are pairs $(T \to X, \sigma)$, where T is a scheme over X and $\sigma \in \Gamma(T, I^1)$ is a section with boundary $\partial\sigma = \tau|_T$. A morphism in $\mathcal{X}$ is defined to be a pair $(f, \rho) : (T, \sigma) \to (T', \sigma')$, where $f : T \to T'$ is a morphism of schemes over X and $\rho \in \Gamma(T, I^0)$ has boundary $\partial\rho = \sigma - f^*(\sigma')$. Given morphisms $(f, \rho) : (T, \sigma) \to (T', \sigma')$, and $(f', \rho') : (T', \sigma') \to (T'', \sigma'')$, the composition is defined to be $(f' \circ f, \rho + f^*(\rho'))$.

We leave it to the reader to check that the forgetful functor $p : \mathcal{X} \to Sch$ makes $\mathcal{X}$ into a category fibred in groupoids over the category of schemes. In fact, for $T \to T'$ there is a natural pullback functor $\mathcal{X}_{T'} \to \mathcal{X}_T$ coming from the restriction maps on the sheaves I^j. To show that $\mathcal{X}$ is a stack for the fppf topology, you use that the I^j are sheaves for the fppf topology. To show that $\mathcal{X}$ is an Artin algebraic stack we find a presentation $U \to \mathcal{X}$. Namely, let $U \to X$ be an étale surjective morphism such that α restricts to zero on U. Thus we can find a $\sigma \in \Gamma(U, I^1)$ whose boundary is τ. The result is a smooth surjective morphism $U \to \mathcal{X}$. Details left to the reader (the fibre product $U \times_{\mathcal{X}} U$ is a $\mathbb{G}_m$-torsor over $U \times_X U$).

2.6 By construction, for any object (T, σ) the autmorphism sheaf $\underline{\mathrm{Aut}}_T(\sigma)$ of σ on Sch/T is identified with $\mathbb{G}_{m,T}$. Namely, it is identified with the sheaf of pairs (id, u), where u is a section of $\mathbb{G}_m$.

2.7 There is a general notion of a quasi-coherent sheaf on an algebraic Artin stack. In our case a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is given by a quasi coherent sheaf $\mathcal{F}_\sigma$ on T for every object (T, σ) of $\mathcal{X}$, and an isomorphism $i(\rho) : f^*\mathcal{F}'_{\sigma'} \to \mathcal{F}$ for every morphism $(f, \rho) : (T, \sigma) \to (T', \sigma')$ of $\mathcal{X}$. These data are subject to the condition $i(\rho' + f^*(\rho)) = f^*(i(\rho')) \circ i(\rho)$ in case of a composition of morphisms as above.

In particular, any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ comes equipped with an action of $\mathbb{G}_{m,\mathcal{X}}$. Namely, the sheaves $\mathcal{F}_\sigma$ are endowed with the endomorphisms $i(u)$. (More precisely we should write $i((\mathrm{id}, u))$.)

2.8 By the above a quasi-coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ can be written canonically as a direct sum

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}^{(m)}.$$

Namely, the summand $\mathcal{F}^{(m)}$ is the piece of $\mathcal{F}$ where the action of $\mathbb{G}_m$ is via the character $\lambda \mapsto \lambda^m$. This follows from the representation theory of the group scheme $\mathbb{G}_m$. See ??.

2.9 The alternative definition of an α-twisted sheaf we mentioned above is a quasi-coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^{(1)}$.In case both definitions make sense they lead to equivalent notions.

2.10 Lemma. *Suppose α is given by a Čhech cocylce. There is an equivalence of the category of α-twisted sheaves with the category of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^{(1)}$.*

Proof. Namely, suppose that α is given by the cocycle α_{ijk} as in Subsection 2.3 and by the section τ of I^2 as in Subsection 2.5. This means that on each U_i we can find a section σ_i with $\partial\sigma_i = \tau$, on each U_{ij} we can find ρ_{ij} such that

$$\partial\rho_{ij} = \sigma_i|_{U_{ij}} - \sigma_j|_{U_{ij}}$$

and that

$$\alpha_{ijk} = \rho_{ij}|_{U_{ijk}} + \rho_{jk}|_{U_{ijk}} - \rho_{ik}|_{U_{ijk}}$$

in $\Gamma(U_{ijk}, I^0)$.

Let us show that a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ such that $\mathcal{F} = \mathcal{F}^{(1)}$ gives rise to an α-twisted sheaf. The reverse construction will be left to the reader. For each i the pair $(U_i \to X, \sigma_i)$ is an object of $\mathcal{X}$. Hence we get a quasi-coherent module $\mathcal{M}_i = \mathcal{F}_{\sigma_i}$ on U_i. For each pair (i, j) the element ρ_{ij} defines a morphism $(\mathrm{id}, \rho_{ij}) : (U_{ij}, \sigma_i|_{U_{ij}}) \to (U_{ij}, \sigma_j|_{U_{ij}})$. Hence, $i(\mathrm{id}, \rho_{ij})$ is an isomorphism we write as $\varphi_{ij} : \mathcal{M}_i \otimes \mathcal{O}_{U_{ij}} \to \mathcal{M}_j \otimes \mathcal{O}_{U_{ij}}$. Finally, we have to check the "α-twisted" cocycle condition of 2.3. The point here is that the equality $\alpha_{ijk} + \rho_{ik}|_{U_{ijk}} = \rho_{ij}|_{U_{ijk}} + \rho_{jk}|_{U_{ijk}}$ implies that $\varphi_{jk} \circ \varphi_{ij}$ differs from φ_{ik} by the action of α_{ijk} on $\mathcal{M}_k$. Since $\mathcal{F} = \mathcal{F}^{(1)}$ this will act on $(U_{ijk} \to U_i)^*\mathcal{M}_i$ as multiplication by α_{ijk} as desired. □

2.11 Generalization. Suppose we have α and β in $H^2(X, \mathbb{G}_m)$. We can obviously define a $\mathbb{G}_m \times \mathbb{G}_m$-gerb $\mathcal{X}_{\alpha,\beta}$ with class (α, β). Every quasi-coherent sheaf $\mathcal{F}$ will have a $\mathbb{Z}^2$ grading. The $(1, 0)$ graded part will be a α-twisted sheaf and the $(0, 1)$ graded part will be a β-twisted sheaf. More generally, the (a, b) graded part is an $a\alpha + b\beta$-twisted sheaf. On $\mathcal{X}_{\alpha,\beta}$ we can tensor quasi-coherent sheaves. Thus we see deduce that there is a tensor functor $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes \mathcal{G}$ which takes as input an α-twisted sheaf and a β-twisted sheaf and produces an $\alpha + \beta$-twisted sheaf. (This is also easily seen using cocyles.) Similarly $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{H}om(\mathcal{F}, \mathcal{G})$ produces an $\beta - \alpha$ twisted sheaf.

In particular a 0-twisted sheaf is just a quasi-coherent $\mathcal{O}_X$-module. Therefore, if $\mathcal{F}, \mathcal{G}$ are α-twisted sheaves, then the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is an $\mathcal{O}_X$-module.

2.12 On a Noetherian algebraic Artin stack any quasi-coherent sheaf is a direct limit of coherent sheaves. See ??. In particular we see the same holds for α-twisted sheaves.

2.13 Azumaya algebras and α-twisted sheaves. Suppose that $\mathcal{F}$ is a finite locally free α-twisted sheaf. Then $A = \mathcal{H}om(\mathcal{F}, \mathcal{F})$ is a sheaf of $\mathcal{O}_X$-algebras. Since it is étale locally isomorphic to the endormophisms of a finite locally free module, we see that A is an Azumaya algebra. It is easy to verify that the Brauer class of A is α.

Conversely, suppose that A is an Azumaya algebra over X. Consider the category $\mathcal{X}(A)$ whose objects are pairs $(T, \mathcal{M}, j)$, where $T \to X$ is a scheme over X, $\mathcal{M}$ is a finite locally free $\mathcal{O}_T$-module, and j is an isomorphism $j : \mathcal{H}om(\mathcal{M}, \mathcal{M}) \to A|_T$. A morphism is defined to be a pair $(f, i) : (T, \mathcal{M}, j) \to (T', \mathcal{M}', j')$ where $f : T \to T'$ is a morphism of schemes over X, and $i : f^*\mathcal{M}' \to \mathcal{M}$ is an isomorphism compatible with j and j'. Composition

of morphisms are defined in the obvious manner. As before we leave it to the reader to see that $\mathcal{X}(A) \to Sch$ is an algebraic Artin stack. Note that each object $(T, \mathcal{M}, j)$ has naturally $\mathbb{G}_m$ (acting via the standard character on $\mathcal{M}$) as its automorphism sheaf over Sch/T.

Not only is it an algebraic stack, but also the morphism $\mathcal{X}(A) \to X$ presents $\mathcal{X}(A)$ as a $\mathbb{G}_m$-gerb over X. Since gerbes are classified by $H^2(X, \mathbb{G}_m)$ we deduce that there is a unique cohomology class α such that $\mathcal{X}(A)$ is equivalent to the gerb $\mathcal{X}$ constructed in Subsection 2.5. Clearly, the gerb $\mathcal{X}$ carries a finite locally free sheaf $\mathcal{F}$ such that $A = \mathcal{H}om(\mathcal{F}, \mathcal{F})$, namely on $\mathcal{X}(A)$ it is the quasi-coherent module $\mathcal{F}$ whose value on the object $(T, \mathcal{M}, j)$ is the sheaf $\mathcal{M}$ (compare with the description of quasi-coherent sheaves in 2.7). Working backwards, we conclude that α is the Brauer class of A.

The following lemma is a consequence of the discussion above.

2.14 Lemma. *The element $\alpha \in H^2(X, \mathbb{G}_m)$ is in $Br(X)$ if and only if there exists a finite locally free α-twisted sheaf of positive rank.*

2.15 Let us use this lemma to reprove the following result (see Hoobler, Proposition 3): If $\alpha \in H^2(X, \mathbb{G}_m)$ and if there exists a finite locally free morphism $\varphi : Y \to X$ such that $\varphi^*(\alpha)$ ends up in $Br(Y)$, then α in $Br(X)$.

Namely, this means there exists a finite locally free α-twisted sheaf $\mathcal{F}$ of positive rank over Y. Let $\mathcal{Y}$ be the $\mathbb{G}_m$-gerb associated to $\alpha|_Y$ and let us think of $\mathcal{F}$ as a sheaf on $\mathcal{Y}$. Let $\tilde{\varphi} : \mathcal{Y} \to \mathcal{X}$ be the obvious morphism of $\mathbb{G}_m$-gerbs lifting φ. The pushforward $\tilde{\varphi}_*\mathcal{F}$ is the desired flat and finitely presented α-twisted sheaf over X.

2.16 Suppose that $\bar{x}$ is a geometric point of X. Then we can lift the morphism $\bar{x} \to X$ to a morphism $\ell : \bar{x} \to \mathcal{X}$ and all such lifts are isomorphic (not canonically). The *fibre* of a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ at $\bar{x}$ is simply defined to be $\ell^*\mathcal{F}$. (Not the stalk!) Notation: $\mathcal{F} \otimes \kappa(\bar{x})$. This is a functor from coherent α-twisted modules to the category of finite dimensional $\kappa(\bar{x})$-vector spaces.

2.17 Suppose that $s \in X$ is a point whose residue field is finite. Then, similarly to the above, we can find a lift $\ell : s \to \mathcal{X}$, and we define the fibre $\mathcal{F} \otimes \kappa(s) := \ell^*\mathcal{F}$ in the same way.

3 The proof

3.1 The only issue is to show that the map $Br(X) \to Br'(X)$ is surjective. Thus let us assume that $\alpha \in H^2(X, \mathbb{G}_m)$ is torsion, say $n\alpha = 0$ for some $n \in \mathbb{N}$.

3.2 As a first step we reduce to the case where X is a quasi-projective scheme of finite type over $\operatorname{Spec} \mathbb{Z}$. This is standard. See 2.1 and 2.2. In particular X is Noetherian, has finite dimension and is a Jacobson scheme.

3.3 Our method does not immediately produce an Azumaya algebra over X. Instead we look at the schemes $X_R = X \otimes R = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$, where $\mathbb{Z} \subset R$ is a finite flat ring extension. We will always assume that R is actually a normal domain. An α-twisted sheaf on X_R means an $\alpha|_{X_R}$-twisted sheaf.

3.4 Lemma. *For any finite set of closed points $T \subset X_R$ there exists a positive integer n and a coherent α-twisted sheaf $\mathcal{F}$ on X_R such that $\mathcal{F}$ is finite locally free of rank n in a neighbourhood of each point $t \in T$.*

Proof. We can find a section $s \in \Gamma(X, \mathcal{L}^N)$ such that the open subscheme X_s is affine and contains the image of T in X. See EGA II, Corollary 4.5.4. By Gabber's result [Hoobler, Theorem 7] we get an Azumaya algebra A over X_s representing α over X_s. Thus we get a finite locally free α-twisted sheaf $\mathcal{F}_s$ of positive rank over X_s, see Subsection 2.14. If $\mathcal{F}_s$ does not have constant rank then we may modify it to have constant rank. (Namely, $\mathcal{F}_s$ will have constant rank on the connected components of X_s. Take suitable direct summands.) Let $j : \mathcal{X}_s \to \mathcal{X}$ be the open immersion which is the pullback of the open immersion $X_s \to X$. Then $j_*\mathcal{F}_s$ is a quasi-coherent α-twisted sheaf. It is the direct limit of its coherent subsheaves, see 2.12. Thus there is a suitable $\mathcal{F} \subset j_*\mathcal{F}_s$ such that $j^*\mathcal{F} \cong \mathcal{F}_s$. The pullback of $\mathcal{F}$ to an α-twisted sheaf over X_R is the desired sheaf. □

3.5 For a coherent α-twisted sheaf $\mathcal{F}$ let $\mathrm{Sing}(\mathcal{F})$ denote the set of points of X_R at which $\mathcal{F}$ is not flat. This is a closed subset of X_R. We will show that by varying R' we can increase the codimension of the singularity locus of $\mathcal{F}$.

3.6 Let $c \geq 1$ be an integer. Induction Hypothesis H_c: For any finite subset of closed points $T \subset X_R$ there exist
(a) a finite flat extension $R \subset R'$, and
(b) a coherent α-twisted sheaf $\mathcal{F}$ on $X_{R'}$
such that
(i) The codimension of $\mathrm{Sing}(\mathcal{F})$ in $X_{R'}$ is $\geq c$,
(ii) the rank of $\mathcal{F}$ over $X_{R'} - \mathrm{Sing}(\mathcal{F})$ constant and positive, and
(iii) the inverse image $T_{R'}$ of T in $X_{R'}$ is disjoint from $\mathrm{Sing}(\mathcal{F})$.

3.7 Note that the case $c = \dim X + 1$ and $T = \emptyset$ implies the theorem in the introduction. Namely, by Subsection 2.14 this implies that α is representable by an Azumaya algebra over $X_{R'}$. By [Hoober, Proposition 3] (see also our 2.15) it follows that α is representable by an Azumaya algebra over X.

3.8 The start of the induction, namely the case $c = 1$, follows easily from Lemma 3.4. Now we assume the hypothesis holds for some $c \geq 1$ and we prove it for $c + 1$.

3.9 Therefore, let $T \subset X$ be a finite subset of closed points. Pick a pair $(R \subset R_1, \mathcal{F}'_1)$ satisfying H_c with regards to the subset $T \subset X$. Set $T_1 = T_{R_1} \cup S'_1$, where $S'_1 \subset \mathrm{Sing}(\mathcal{F}'_1)$ is a choice of a finite subset of closed points with the property that S'_1 contains at least one point from each irreducible component of $\mathrm{Sing}(\mathcal{F}'_1)$ that has codimension c in $X \otimes R_1$. Next, let $(R_1 \subset R_2, \mathcal{F}'_2)$ be a pair satisfying H_c with regards to the subset $T_1 \subset X_{R_1}$. Set $T_2 = (T_1)_{R_2} \cup S'_2$, where $S'_2 \subset \mathrm{Sing}(\mathcal{F}'_2) - \mathrm{Sing}(\mathcal{F}'_1) \otimes R_2$ is a finite subset of closed points which contains at least one point of each irreducible component of $\mathrm{Sing}(\mathcal{F}'_2)$ that has codimension c in $X \otimes R_2$. Such a set S'_2 exists because by construction the irreducible components of codimension c of $\mathrm{Sing}(\mathcal{F}'_2)$ are not contained in $\mathrm{Sing}(\mathcal{F}_1) \otimes R_2$. Choose a pair $(R_2 \subset R_3, \mathcal{F}'_3)$ adapted to T_2. Continue like this until you get a pair $(R_{n+1}, \mathcal{F}'_{n+1})$ adapted to $T_n \subset X_{R_n}$. (Recall that n is a fixed integer such that $n\alpha = 0$.) For clarity, we

stipulate at each stage that

$$S_j \subset \mathrm{Sing}(\mathcal{F}'_j) - \bigcup_{i<j} \mathrm{Sing}(\mathcal{F}'_i) \otimes R_j,$$

contains at least one point from each irreducible component of $\mathrm{Sing}(\mathcal{F}'_j)$ that has codimension c in $X \otimes R_j$. We also choose a set S_{n+1} like this.

Let us write $\mathcal{F}_i = \mathcal{F}'_i \otimes_{R_i} R_{n+1}$, and $S_i = S'_i \otimes_{R_i} R_{n+1}$. These are coherent α-twisted sheaves on $X \otimes R_{n+1}$ and finite subsets of closed points $S_i \subset X \otimes R_{n+1}$. They have the following properties:

(a) The subset $T \otimes R_{n+1}$ is disjoint from $\mathrm{Sing}(\mathcal{F}_i)$.
(b) Each $\mathcal{F}_i$ has a constant positive rank over $X_{R_{n+1}} - \mathrm{Sing}(\mathcal{F}_i)$.
(c) Each component of $\mathrm{Sing}(\mathcal{F}_i)$ of codimension c in $X \otimes R_n$ meets the subset S_i.
(d) For each $s \in S_i$ the sheaves $\mathcal{F}_j$ are finite locally free at s for $j \neq i$.

From now on we will only use the coherent α-twisted sheaves $\mathcal{F}_i$, $i = 1, \ldots, n+1$, the subsets S_i, $i = 1, \ldots, n+1$ and the properties (a), (b), (c) and (d) above. We will use them to construct an α-twisted sheaf $\mathcal{F}$ over $X_{R'}$ for some finite flat extension $R_{n+1} \subset R'$ whose singularity locus $\mathrm{Sing}(\mathcal{F})$ is contained in $\bigcup \mathrm{Sing}(\mathcal{F}_i) \otimes R'$ such that $S_i \otimes R' \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$. It is clear that this will be a sheaf as required in H_{c+1}, and it will prove the induction step.

3.10 For ease of notation we write R in stead of R_{n+1} from now on, so that the coherent α-twisted sheaves $\mathcal{F}_i$ are defined over X_R, and so that the $S_i \subset X \otimes R$. Note that we may replace $\mathcal{F}_i$ by direct sums $\mathcal{F}_i^{m_i}$ for suitable integers m_i such that each $\mathcal{F}_i$ has the same rank r over the open $X \otimes R - \mathrm{Sing}(\mathcal{F}_i)$. We may also assume that r is a large integer.

3.11 Consider the α-twisted sheaf

$$\mathcal{G}_1 := \mathcal{F}_1^{\oplus r^n} \oplus \ldots \oplus \mathcal{F}_{n+1}^{\oplus r^n}$$

and since $(n+1)\alpha = \alpha$ the α-twisted sheaf

$$\mathcal{G}_2 := \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_{n+1}.$$

Consider also the sheaf of homomorphisms

$$\mathcal{H} := \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$$

which is a coherent $\mathcal{O}_{X \otimes R}$-module (see 2.11). Recall that $\mathcal{L}$ is our ample invertible sheaf. Since $X_R \to X$ is finite, $\mathcal{L}_R$ is ample on $X \otimes R$ as well. For a very large integer N we are going to take a section ψ of the space

$$\Gamma_N := \Gamma(X \otimes R, \mathcal{H} \otimes \mathcal{L}^N) = \Gamma\big(X \otimes R, \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2 \otimes \mathcal{L}^N)\big).$$

The idea of the proof is that for a general ψ as above the kernel of the map of α-twisted sheaves

$$\mathcal{G}_1 \to \mathcal{G}_2 \otimes \mathcal{L}^N$$

is a solution to the problem we are trying to solve. However, it is not so easy to show there are any "general" sections. This is why we may have to extend our ground ring R a little.

3.12 Claim. *The kernel of a $\psi \in \Gamma_N$ is a solution to the problem if it satisfies the following properties:*

(a) For every geometric point $\bar{x} \in X$, $\bar{x} \notin \bigcup \mathrm{Sing}(\mathcal{F}_i)$ the map

$$\psi_{\bar{x}} : \mathcal{G}_1 \otimes \kappa(\bar{x}) \to \mathcal{G}_2 \otimes \mathcal{L}^N \otimes \kappa(\bar{x})$$

is a surjection. (See 2.16 for the fibre functor.)

(b) Let $s \in S_i$ for some i. Then the composition

$$\mathcal{F}_i^{r^n} \otimes \kappa(s) \to \mathcal{G}_1 \otimes \kappa(s) \to \mathcal{G}_2 \otimes \mathcal{L}^N \otimes \kappa(s)$$

is an isomorphism. (See 2.17.)

We will study the sheaf $\mathcal{H}$ and its sections. Along the way we will show that the claim holds and we will show that there is a finite extension $R \subset R'$ such that a section $\psi \in \Gamma_N \otimes R'$ can be found satisfying (a), (b) above. This will prove the theorem.

3.13 Local study of the sheaf $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$. Let $\mathrm{Spec}(B) \to X \otimes R$ be a morphism of schemes. Assume that $\alpha|_{\mathrm{Spec}(B)} = 0$. Thus there is a 1-morphism $\ell : \mathrm{Spec}(B) \to \mathcal{X}$ lifting $\mathrm{Spec}(B) \to X$. In other words the sheaves $\mathcal{F}_i$ give rise to finitely generated B-modules M_i. Then $\mathcal{G}_1$ corresponds to

$$M_1^{r^n} \oplus \ldots \oplus M_{n+1}^{r^n}$$

and $\mathcal{G}_2$ corresponds to

$$M_1 \otimes_B M_2 \otimes_B \ldots \otimes_B M_{n+1}.$$

Finally, the sheaf $\mathcal{H} = \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ pulls back to the quasi-coherent sheaf associated to the B-module

$$\mathrm{Hom}_B(M_1^{r^n} \oplus \ldots \oplus M_{n+1}^{r^n}, M_1 \otimes_B M_2 \otimes_B \ldots \otimes_B M_{n+1}).$$

3.14 In particular, if $x \in X$ is a point and $\mathcal{O}_{X,x} \subset B$ is an étale local ring extension, then the stalk $\mathcal{H}_x$ is an $\mathcal{O}_{X,x}$-module $\mathcal{H}_x$ such that $\mathcal{H}_x \otimes B$ is isomorphic to the module displayed above.

3.15 Suppose that the point x is not in any of the closed subsets $\mathrm{Sing}(\mathcal{F}_i)$. Then, with the notation above, we can choose isomorphisms $M_i = B^r$ for all i, and we see that

$$\mathcal{H}_x \otimes B = \mathrm{Hom}_B(B^{(n+1)r^{n+1}}, B^{r^{n+1}}) \cong \mathrm{Mat}((n+1)r^{n+1} \times r^{n+1}, B).$$

We conclude that the condition (a) at a geometric point $\bar{x}$ over x is defined by a cone $C(\bar{x})$ in $\mathcal{H} \otimes \kappa(\bar{x})$ of codimension $(n+1)r^{n+1} - r^{n+1} + 1 \geq nr^n$.

3.16 Let H be the vector bundle over $U := X - \bigcup \mathrm{Sing}(\mathcal{F}_i)$ whose sheaf of sections is $\mathcal{H}|_U$. (So H is the spectrum of the symmetric algebra on the dual of $\mathcal{H}$.) The description above shows that there is a cone

$$C \subset H$$

whose fibre at each point is the "forbidden" cone of lower rank maps. (In particular the codimension of the fibres $C_x \subset H_x$ is large.) Namely, the description above defines this cone étale locally over U. But the rank condition describing C is clearly preserved by the gluing data and hence C descends.

3.17 Suppose that the point x (as in 3.14) is one of the closed points $s \in S_i$. In this case we can find an extension $\mathcal{O}_{X,x} \subset B$ as in 3.14 with trivial residue field extension. Namely, the restriction of α to the henselization $\mathcal{O}^h_{X,s}$ will be trivial. Namely, the Brauer group of a henselian local ring is easily seen to be equal to the Brauer group of the residue field which is trivial in this case. In this case the modules M_j for $j \neq i$ are finite locally free, but M_i is not. We choose isomorphisms $M_j = B^r$ for $j \neq i$ and we set $M_i = M$. Thus we observe that in this case

$$\mathcal{H}_s \otimes B = \mathrm{Hom}_B(M^{r^n}, M^{r^n}) \oplus \mathrm{Hom}_B(B^{nr^{n+1}}, M^{r^n}).$$

The condition (b) for our point s simply means that if we write $\psi \otimes B = \psi_1 \oplus \psi_2$, then $\psi_1 \otimes \kappa(s)$ should be a surjection from $M^{r^n} \otimes \kappa(s)$ to itself. By Nakayama's Lemma and the fact that a surjective self map of a finitely generated module over a Noetherian local ring is an isomorphism, this is equivalent to the condition that ψ_1 is an isomorphism.

3.18 Let $\psi_s \in \mathcal{H} \otimes \kappa(s)$ be an element which corresponds to the image $\mathrm{id}_{M^{r^n}} \oplus 0$ in the fibre under an isomorphism as above. Fix an isomorphism $\mathcal{L} \otimes \kappa(s) = \kappa(s)$ so that we can identify the fibre of $\mathcal{H}$ and $\mathcal{H} \otimes \mathcal{L}^N$. The following observation will be used below: If we have a global section $\psi \in \mathcal{H} \otimes \mathcal{L}^N$ which reduces modulo the maximal ideal $\mathfrak{m}_s$ to a nonzero multiple of ψ_s for every $s \in S_i$ and every i then ψ has property (b) of 3.12.

3.19 Proof of Claim 3.12. Suppose that ψ satisfies (a) and (b). Over the open subscheme U we see that $\ker(\psi)$ is the kernel of a surjective map of finite locally free α-twisted sheaves. Hence $\ker(\psi)$ is finite locally free over U. At each point $s \in S$ we see that $\ker(\psi) \otimes B$ is going to be isomorphic to the factor $B^{nr^{n+1}}$ of $\mathcal{G}_{1,B}$. Thus it is finite locally free at each s as well.

3.20 As a last step we still have to show that there exists a section ψ satisfying (a), (b) of 3.12 possibly after enlarging R. This will occupy the rest of the paper. It has nothing to do with Brauer groups. In fact, according to Subsection 3.18, we can formulate the problem as follows.

3.21 Situation. Let X be a quasi-projective scheme of finite type over R finte flat over $\mathbb{Z}$ as above. Let $\mathcal{L}$ be an ample invertible sheaf. Suppose that $\mathcal{H}$ is a coherent $\mathcal{O}_X$-module which is finite locally free over an open subscheme $U \subset X$. Let H be the vector bundle U whose sheaf of sections is $\mathcal{H}|_U$. Suppose that $C \subset H$ is closed cone in H. Let $c \in \mathbb{N}$ be such that the codimension of C_u in H_u is bigger than c for all $u \in U$. Furthermore, suppose $S \subset X$ is a finite set of closed points not in U, and suppose that we are given "prescribed values" $\psi_s \in \mathcal{H} \otimes \kappa(s)$ (compare with 3.18).

3.22 Claim. *In the situation above, suppose that $c > \dim X + 1$. There exists an N and a finite flat extension $R \subset R'$ and a section ψ of $\mathcal{H} \otimes \mathcal{L}^N \otimes_R R'$ such that:*

(a) The set of points u of $U \otimes_R R'$ such that $\psi_u \in C_u \otimes L_u^{\otimes N}$ is empty.

(b) For each closed point s' of $X_{R'}$ lying over a point of $s \in S$, the value of $\psi \otimes \kappa(s')$ of ψ at s' is a nonzero multiple of ψ_s. (Where the multiplier is actually an element of $L^N \otimes \kappa(s')$.)

Proof. First we remark that it suffices to prove the claim for $R = \mathbb{Z}$. Namely, given a situation 3.21 and a solution ψ relative to $X/\mathbb{Z}$ on $X \otimes_{\mathbb{Z}} R''$ then we just set $R' = R \otimes_{\mathbb{Z}} R''$. So now we assume that $R = \mathbb{Z}$.

Choose N_0 so that for all $N \geq N_0$ there is a finite collection of global sections

$$\{\Psi_i \in \Gamma(X, \mathcal{I}_S \otimes \mathcal{H} \otimes \mathcal{L}^N), i \in I\}$$

such that the map

$$\bigoplus_{i \in I} \mathcal{O}_X \xrightarrow{\sum \Psi_i} \mathcal{I}_S \otimes \mathcal{H} \otimes \mathcal{L}^N$$

is surjective. For each $s \in S$ we may choose a global section

$$\Psi_s \in \Gamma(X, \mathcal{I}_{S-\{s\}} \otimes \mathcal{H} \otimes \mathcal{L}^N)$$

which reduces to a nonzero scalar multiple of ψ_s at s (again we may need to increase N). Let $\mathbb{A} = \operatorname{Spec} \mathbb{Z}[x_i, x_s; i \in I, s \in S]$. There is a universal section

$$\Psi = \sum x_i \Psi_i + \sum x_s \Psi_s$$

of the pull back of $\mathcal{H} \otimes \mathcal{L}^N$ to $\mathbb{A} \times_{\operatorname{Spec} \mathbb{Z}} X$. In particular, in a point (a, u) of $\mathbb{A} \times U \subset \mathbb{A} \times X$ the fibre $\Psi_{(a,u)}$ can be seen as a point of $H_u \otimes L_u^N$. In the open subscheme $\mathbb{A} \times U$ we let Z be the closed subset described by the following formula:

$$Z = \{(a, u) \mid \Psi_{(a,u)} \in C_u \otimes L_u^N\}$$

The fibres of the morphism $Z \to U$ have dimension at most $\#S + \#I - c$, by our assumption that c bounds the codimensions of the cones C_u from below, and the property that the sections Ψ_i generate the sheaf $\mathcal{H} \otimes \mathcal{L}^N$. Thus the dimension of Z is at most $\dim \mathbb{A} - 2$ (note that the dimension of $\mathbb{A}$ is $\#S + \#I + 1$).

Let $\bar{Z}$ be the closure of the image of Z in $\mathbb{A}$. For each element $s \in S$, let $t_s \in \operatorname{Spec} R$ be the image of s in $\operatorname{Spec} R$ under the morphism $X \to \operatorname{Spec} R$. Let $Z_s \subset \mathbb{A}_t$ be the closed subscheme defined by $x_s = 0$. This has codimension 2 in $\mathbb{A}$ as t is a closed point of $\operatorname{Spec} R$.

We want an $\mathbb{Z}$-rational point of $\mathbb{A}$ which avoids $\bar{Z} \cup \bigcup Z_s$. This may not be possible. However, according to the main result of [1] we can find a finite extension R of $\mathbb{Z}$ and morphism $\operatorname{Spec} R$ into $\mathbb{A}$ which avoids $\bar{Z} \cup \bigcup Z_s$ as desired.

References

[1] Moret-Bailly, Laurent, *Groupes de Picard et problmes de Skolem. I, II,* Ann. Sci. cole Norm. Sup. (4) **22** (1989), no. 2, pp. 161–179, pp. 181–194.