RATIONAL CURVES ON FERMAT HYPERSURFACES

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ABSTRACT. In this note we study rational curves on degree p^r+1 Fermat hypersurface in $\mathbb{P}_k^{p^r+1}$, where k is an algebraically closed field of characteristic p. The key point is that the presence of Frobenius morphism makes the behavior of rational curves to be very different from that of charateristic 0. We show that if there exists N_0 such that for all $e \geq N_0$ there is a degree e very free rational curve on X, then $N_0 > p^r(p^r-1)$.

1. INTRODUCTION

Rational curves appear to be very important in the study of higher dimensional algebraic varieties. We refer to [Ko] for the background. Let X be a smooth projective variety over an algebraically closed field k.

Definition 1.1. A rational curve $f: C \cong \mathbb{P}^1 \to X$ is free (resp. very free) if f^*T_X is globally generated (resp. ample). We say that X is *separably rationally connected* (SRC) if there is a very free rational curve on X.

Definition 1.2. X is *rationally connected* (RC) if a general pair of points can be connected by a rational curve. This means that there is a family of rational curves $\pi: U \to Y$ together with a morphism $u: U \to X$ such that the natural map $u^{(2)}: U \times_Y U \to X \times_k X$ is dominant. If we only require the general fiber of π to be a genus 0 curve, then we say that X is rationally chain connected.

One very important tool to study rational curves is deformation theory. This works especially well in characteristic 0. For example, it is easy to see that SRC implies RC in any characteristic. But if the characteristic is 0, the then RC is equivalent to SRC. One very important class of rationally connected varieties is provided by the following

Theorem 1.3. *([KMM],[Ca]) Smooth Fano varieties over a field of characteristic 0 are rationally connected.*

The case of characteristic p is still mystery. We know that all smooth Fano varieties are rationally chain connected, see V.2 of [Ko]. Kollár has constructed examples of singular Fano's that are not SRC, see V.5 of [Ko]. This naturally leads to the question whether all smooth Fano varieties are SRC. Recently Y. Zhu has proved that a general Fano hypersurface is SRC, see [Zh].

In this note, we consider a class of very special Fano hypersurfaces over a field of positive characteristic. From now on, we fix k to be an algebraically closed field of characteristic p. Let $X = X_{d,N} \subset \mathbb{P}_k^N$ be the Fermat hypersurface defined by

$$X_0^d + X_1^d + \cdots + X_N^d = 0,$$

where $d = p^r + 1$ and $N \geq d$. We ask the following

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Question 1.4. Is there a very free rational curve on X?

Lemma 1.5. *If $X_{d,N}$ contains a very free rational curve for $N = p^r + 1$, then $X_{d,N}$ contains a very free rational curve for all $N \geq p^r + 1$.*

By this Lemma, we see that the most interesting case is when $d = N$. Our first observation is

Proposition 1.6. *Let $X = X_{d,d}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^d_k$ and M_e be the space of degree e morphisms from $\mathbb{P}^1$ to X. Then for M_e to have the expected dimension, e has to be at least $p^r - 1$. In particular, if $e < p^r - 1$ then there is no free rational curve of degree e.*

For very free rational curves on $X_{d,d}$ we have the following

Theorem 1.7. *Let $X = X_{d,d}$ be the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbb{P}^d_k$. Let $f : C = \mathbb{P}^1 \to X$ be a rational curve of degree e. If $mN < e \leq (m+1)(N-1)$ for some $0 \leq m \leq N - 3$, then f is not very free.*

Corollary 1.8. *If there exists N_0 such that for all $e \geq N_0$ there is a very free rational curve of degree e on X, then $N_0 > p^r(p^r - 1)$.*

Definition 1.9. A *rational normal curve* $f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ is a rational curve of degree N whose linear span is the whole $\mathbb{P}^N$.

One potential candidate of a very free rational curve of low degree on X is given by the following

Proposition 1.10. *Let $X = X_{d,N}$ be as above. If $C \subset X$ is a rational normal curve (viewed as a rational curve on $\mathbb{P}^N$), then C is very free on X.*

One of the simplest unknown cases is the following question.

Question 1.11. Let X be the degree 5 Fermat hypersurface in $\mathbb{P}^5$ over $\overline{\mathbb{F}}_2$. Is there a rational normal curve on X?

2. Proofs

Let $X = X_{d,N}$. We use the following diagram to investigate the tangent sheaf of X.

(1)
$$
\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & T_X \arrow[r] \arrow[u] & T_{\mathbb{P}^N}|_X \arrow[r] \arrow[u] & \mathcal{O}_X(p^r+1) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[u] & \mathcal{O}_X(1)^{N+1} \arrow[r, "(X_i^{p^r})"] \arrow[u] & \mathcal{O}_X(p^r+1) \arrow[r] \arrow[u, equal] & 0 \\
 & \mathcal{O}_X \arrow[r, equal] \arrow[u] & \mathcal{O}_X \arrow[u, "(X_i)"] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

We dualize the second column and get

$$0 \longrightarrow \Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X^{N+1} \xrightarrow{(X_i)} \mathcal{O}_X(1) \longrightarrow 0$$

Let $F : X \to X$ be the Frobenius morphism. We apply $(F^*)^r$ to the above exact sequence and get

$$0 \longrightarrow (F^*)^r\Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_X(p^r) \longrightarrow \mathcal{O}_X^{N+1} \xrightarrow{(X_i^{p^r})} \mathcal{O}_X(p^r) \longrightarrow 0 \tag{2}$$

Compare this with the second row of (1), we get

$$\mathscr{F} \cong (F^*)^r\Omega^1_{\mathbb{P}^N} \otimes \mathcal{O}_X(p^r+1) \tag{3}$$

Proof. (of Proposition 1.10). Since $f : C \to \mathbb{P}^N$ is a rational normal curve, we have

$$f^*\Omega^1_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^1}(-N-1)^{\oplus N}$$

Hence

$$f^*\mathscr{F} = \mathcal{O}_{\mathbb{P}^1}((-N-1)p^r + (p^r+1)N)^{\oplus N} = \mathcal{O}_{\mathbb{P}^1}(N-p^r)^N$$

is very ample. The first column of diagram (1) shows that f^*T_X is very ample. □

Proof. (of Theorem 1.7). Consider the first column in diagram (1), we know that if the splitting of $f^*\mathscr{F}$ has a negative summand or has at least two copies of $\mathcal{O}_{\mathbb{P}^1}$, then f^*T_X is not ample. To avoid this from happening, the best situation is when $\Omega^1_{\mathbb{P}^N}|_C$ is balanced. Now we assume that all the above splittings are balanced. Let $a = [\frac{(N+1)e}{N}] = e + [\frac{e}{N}]$. Then

$$\Omega^1_{\mathbb{P}^N}|_C \cong \mathcal{O}(-a)^l \oplus \mathcal{O}(-a-1)^{l'}$$

with $l' = (N+1)e - Na$ and $l = N - l'$. Then it follows from (3) that

$$f^*\mathscr{F} \cong \mathcal{O}(b_1)^l \oplus \mathcal{O}(b_2)^{l'}$$

with $b_1 = -ap^r + e(p^r+1)$ and $b_2 = -(a+1)p^r + e(p^r+1)$. Note that $f^*\mathscr{F}$ is highly unbalance unless e is a multiple of N. If $mN < e < (m+1)N$, then $a = e + m$ and

$$b_2 = -(a+1)p^r + e(p^r+1) = e - (m+1)(N-1)$$

Hence we have $b_2 < 0$ if $e < (m+1)(N-1)$. If $e = (m+1)(N-1)$, then

$$l' = (N+1)e - Na = N - m - 1$$

The theorem follows easily from this computation. □

Proof. (of Proposition 1.6). A degree e rational curve $f : \mathbb{P}^1 \to X$ can be written as

$$t \mapsto [f_0 : f_1 : \cdots : f_d]$$

where $f_i = \sum_{j=0}^e a_{ij}t^j$ are polynomials of degree at most e. The condition for the image of f to be contained in X is given by $F = \sum f_i^d = 0$ as a polynomial in t. Note that $\deg F = de$, hence we expect to get $de+1$ equations in the coefficients a_{ij}. By dimension count, we easily see that the expected dimension of M_e is $d + e - 1$. By explicit computation we have

$$f_i^d = (\sum_{j=0}^e a_{ij}t^j)^{p^r}(\sum_{j=0}^e a_{ij}t^j) = (\sum_{j=0}^e a_{ij}^{p^r}t^{jp^r})(\sum_{j=0}^e a_{ij}t^j)$$

If $e < p^r - 1$ the coefficient of t^{jp^r-1}, $j = 1, \dots, e$, is automatically 0. Hence the actual dimension of M_e is bigger than the expected dimension. □

Proof. (of Lemma 1.5). It is easy to realize $X_{d,N}$ as hyperplane section of $X_{d,N+1}$. This implies that any very free rational curve on $X_{d,N}$ gives a very free rational curve on $X_{d,N+1}$. □

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