Conics on the Fermat quintic threefold

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University of Oxford

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A *morphism* between two affine varieties is a function given by polynomials in the coordinates of the ambient affine space.

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- we remove 0 from their vanishing set;
- we consider solutions only up to multiplication by a scalar.

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We are therefore naturally led to consider the *projective space* $\mathbb{P}^n$ over k, whose k-points are

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A *projective variety* over k is the set of solutions in $\mathbb{P}^n(k)$ of a system of *homogeneous* polynomials in $n+1$ variables.

A projective variety admits a cover by affine varieties: if x is a variable, then $\mathbb{P}^n \setminus \{x = 0\} \simeq \mathbb{A}^n$.

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A *morphism* between two varieties is an everywhere defined function that is locally a rational function in the coordinates.

Rational curves

To study a variety we are going to search for "parameterized curves" inside it. For example, the image of the morphism

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 \\ t & \longmapsto & (t, t^2) \end{array}$$

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is the affine variety in $\mathbb{A}^2$ defined by the equation $y = x^2$. We can also homogenize the above example:

$$\begin{array}{ccc} t & \longmapsto & (t, t^2) \\ \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 \\ \cap & & \cap \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ [s, t] & \longmapsto & [s^2, st, t^2]. \end{array}$$

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Example 1. Recall the unit circle $S^1 := \{x^2 + y^2 - 1 = 0\} \subset \mathbb{A}^2$.

The "parameterization"

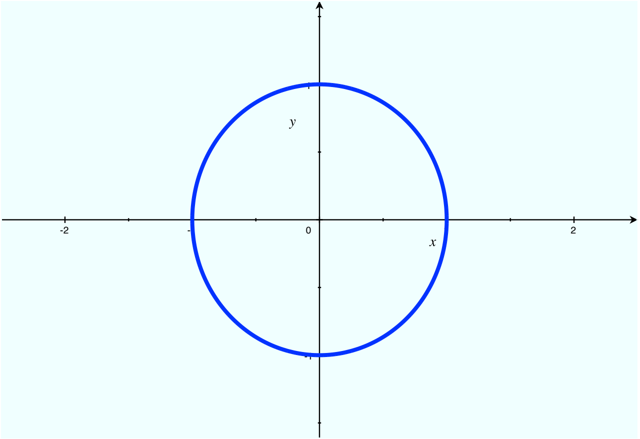
$$\begin{array}{rcl} \mathbb{R} & \longrightarrow & S^1 \\ t & \longmapsto & (\cos(t), \sin(t)) \end{array}$$

of S^1 is **not** in terms of *polynomials* in the coordinates.

We want a polynomial parameterization.

Rational curves

Instead, we proceed using stereographic projection.



This is the circle $S^1 \colon x^2 + y^2 = 1$ over $\mathbb{R}$.

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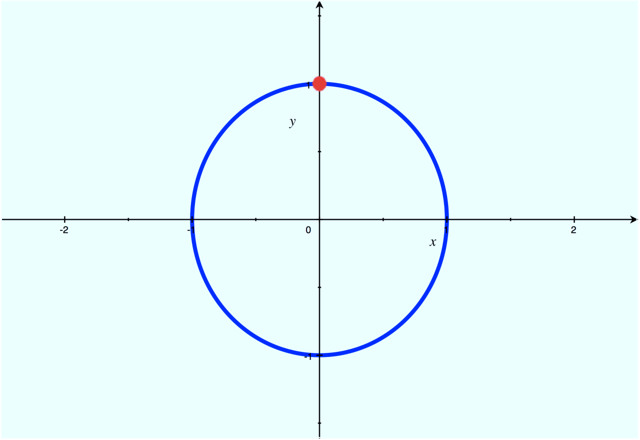

We let P be the point with coordinates $P := (0, 1)$.

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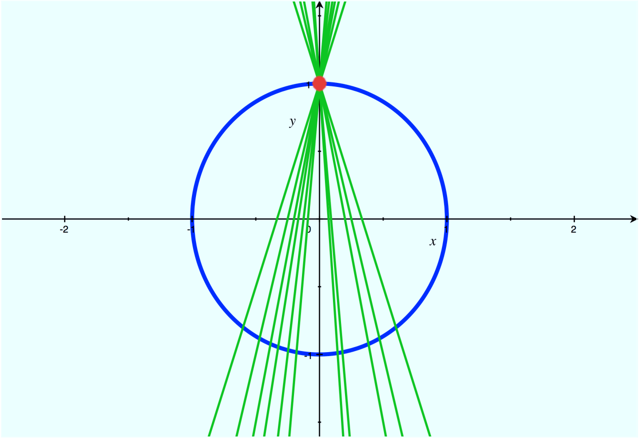

We project from the point P to the x-axis.

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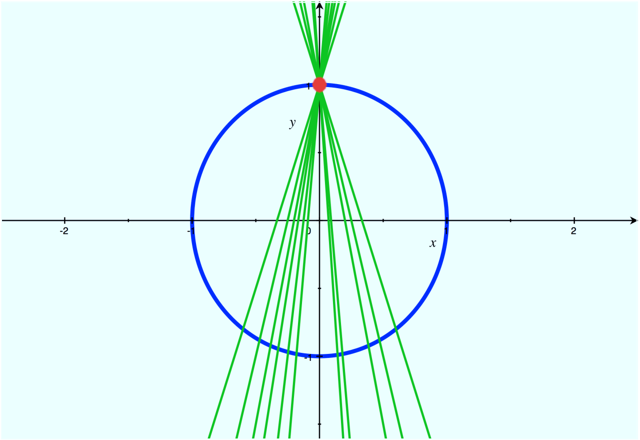

Each line through P intersects the circle S^1 at P and at one more point: the points of S^1 are identified with the lines through P.

Rational curves

Thus we found

$$S^1 \quad \longrightarrow \quad \{\text{lines in } \mathbb{A}^2 \text{ through } P\} \simeq \mathbb{P}^1$$

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More precisely, homogenizing we replace

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$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & C \subset \mathbb{P}^2 \\ [s,t] & \longmapsto & [s^2 - t^2, 2st, s^2 + t^2]. \end{array}$$

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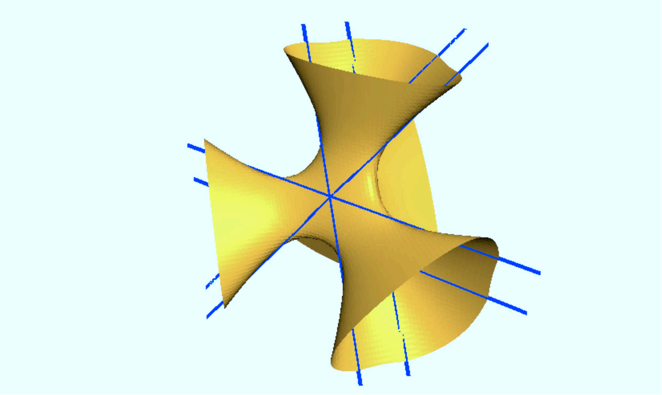
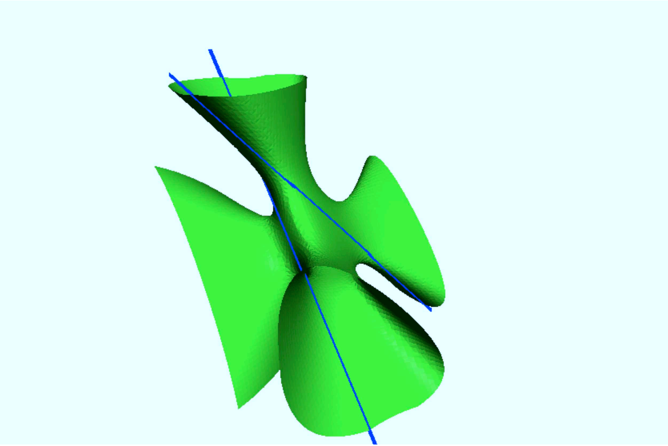
The curve C is a *conic*. Any non-singular conic with a point can be parameterized by the same argument.

Example 2. Let X be a non-singular cubic surface in $\mathbb{P}^3$.

Classical fact: X contains exactly 27 lines, among them there pairs of skew lines (in fact, there are 216 such pairs).

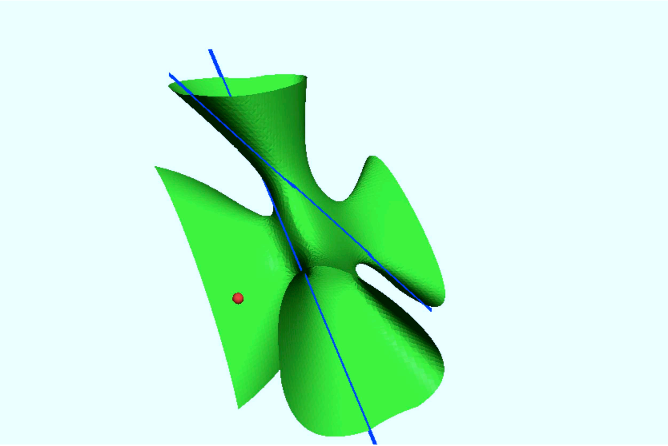


Let L_1, L_2 be two skew lines on the cubic surface $X \subset \mathbb{P}^3$.



Rational curves

For each point $P \in X$ let $\varphi_1(P) \in L_1$ be the point of intersection of the line L_1 with the plane containing L_2 and the point P.



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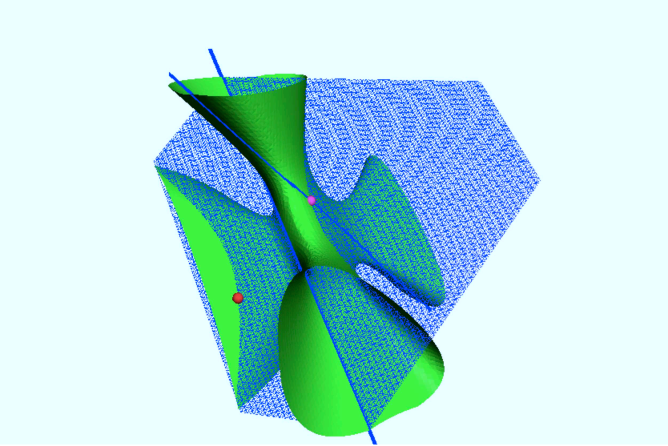
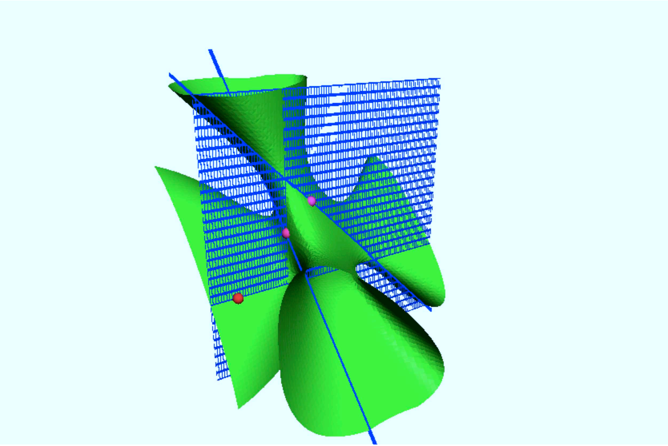


Similarly, let $\varphi_2(P) \in L_2$ be the point of intersection of the line L_2 with the plane containing L_1 and the point P.



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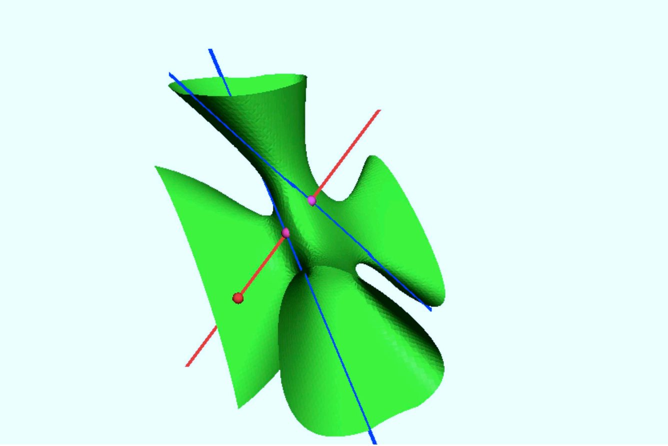
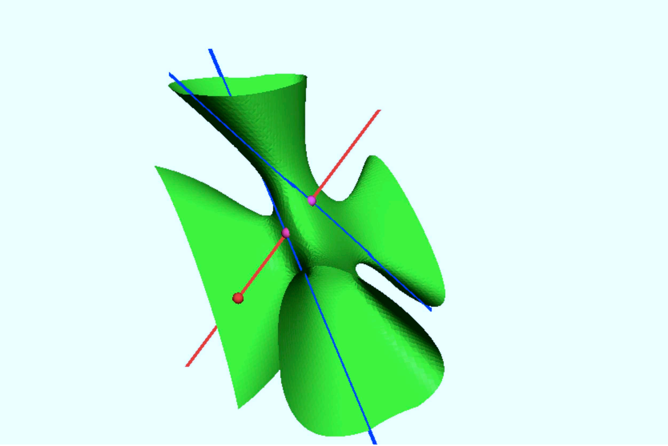


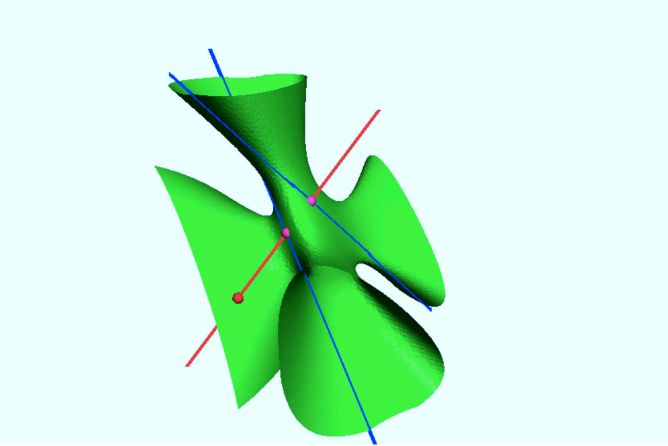
We constructed a morphism

$$(\varphi_1, \varphi_2) \colon X \longrightarrow L_1 \times L_2 \ \simeq \ \mathbb{P}^1 \times \mathbb{P}^1$$

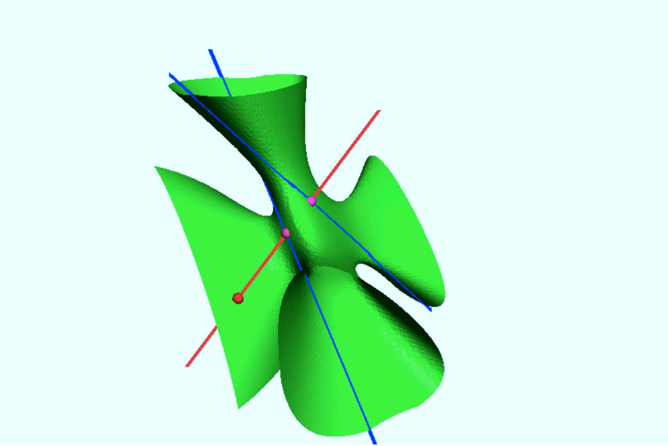
admitting an inverse almost everywhere.



"Inverse": $(p_1, p_2) \in L_1 \times L_2$ determines uniquely a line $L_{12} \subset \mathbb{P}^3$; apart from p_1, p_2 the line L has one more intersection point p_{12} with X. The assignment $(p_1, p_2) \mapsto p_{12}$ is the required inverse.



Remark. Any cubic surface is isomorphic to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at five points, and also to the blow up of $\mathbb{P}^2$ at six points.



Recap

Conics are rational curves, and can be parameterized if one of their points is known.

At least in the case of cubic surfaces, the knowledge of their lines can be exploited to explicitly determine how they look like.

Remark. Smooth cubic surfaces also contain conics: any plane containing a line intersects the cubic in the line and a residual conic.

Quintic threefolds

We move on to *quintic threefolds* in $\mathbb{P}^4$.

Facts on smooth quintic threefolds

Hodge diamond	Betti numbers
1	1
$0 \quad 0$	0
$0 \quad 1 \quad 0$	1
$1 \quad 101 \quad 101 \quad 1$	204
$0 \quad 1 \quad 0$	1
$0 \quad 0$	0
1	1

They are Calabi-Yau threefolds.
They are of interest in string theory and are a good testing ground for Mirror Symmetry.

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A general quintic threefold contains exactly 2,875 lines and exactly 609,250 conics.

There is a formula for the *number of rational curves of degree d* on a general quintic threefold, even though it is not known in general that there are only finitely many rational curves!

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The ψ-pencil above is called the *Dwork pencil* and it is often studied for its relation to Mirror Symmetry.

Fermat quintic threefold

Setting $\psi = 0$ in the Dwork pencil, we obtain the *Fermat quintic*

$$X\colon \{x^5 + y^5 + z^5 + u^5 + v^5 = 0\} \subset \mathbb{P}^4.$$

The Fermat quintic is the focus of our attention from now on.

Most of what we say works more generally on every quintic in the Dwork pencil.

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$$(x^5 + y^5 + z^5) + \underbrace{(u^5 + v^5)}_{\substack{\text{factor} \\ \text{this term}}}$$

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A point $[x_0, y_0, z_0]$ on the Fermat quintic curve $x^5+y^5+z^5=0$ determines the five lines

$$\left\{\ [sx_0\,,\ sy_0\,,\ sz_0\,,\ \zeta t\,,\ t]\ :\ [s,t]\in\mathbb{P}^1\ \right\}_\zeta$$

on X, one for each ζ with $\zeta^5-1=0$. We find 50 one-parameter families of lines on X. There are no other lines on X.

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A short computer search shows that there plenty of conics in X modulo every prime less than 40.

In fact, there are so many conics, that we suspect that there might be a one parameter family of them!

So far, though, we only found conics over finite fields.

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- for some primes, the conics pair up in **coplanar** pairs;
- even for the primes where no pairing occurs, there still are some coplanar pairs;
- clearly, a plane containing two conics of X has a residual intersection with X consisting of a line.

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The strategy is to determine conditions on α and β so that the plane described above contains two conics of X.

Intersect X with the two-dimensional linear subspace

$$\begin{cases} x & = & \alpha v + s \\ y & = & \alpha v - s \\ z & = & \beta v + t \\ u & = & \beta v - t, \end{cases}$$

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where $\alpha, \beta \in k$ and s, t are parameters. We find

$$v\Big((1+2\alpha^5+2\beta^5)v^4+20\alpha^3v^2s^2+20\beta^3v^2t^2+10\alpha s^4+10\beta t^4\Big)=0$$

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The component $v=0$ corresponds to the line in X

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If everything works out, with a careful choice of α and β, the intersection of X with the plane above consists of the line $v=0$ and of a union of two conics.

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The component where v is not identically zero is defined by the factor of degree four in v, s, t *involving only even degree exponents*!

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We rewrite it in terms of $V := v^2$, $S := s^2$ and $T := t^2$

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so that it defines a conic in the V, S, T plane.

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and determinant

$$\Delta(\alpha, \beta) := 800\alpha\beta(1 - 8\alpha^5 - 8\beta^5).$$

$$(1 + 2\alpha^5 + 2\beta^5)V^2 + 20\alpha^3 VS + 20\beta^3 VT + 10\alpha S^2 + 10\beta T^2 = 0$$

This is the quadratic form associated to the bilinear form

$$\begin{pmatrix} 2(1 + 2\alpha^5 + 2\beta^5) & 20\alpha^3 & 20\beta^3 \\ 20\alpha^3 & 20\alpha & 0 \\ 20\beta^3 & 0 & 20\beta \end{pmatrix}$$

and determinant

$$\Delta(\alpha, \beta) := 800\alpha\beta(1 - 8\alpha^5 - 8\beta^5).$$

The quadratic form in V, S, T is a product of linear forms if

$$\Delta(\alpha, \beta) = 0.$$

The determinant $\Delta(\alpha, \beta) = 800\alpha\beta(1 - 8\alpha^5 - 8\beta^5)$ vanishes

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We exclude the cases $\alpha\beta = 0$.

Summarizing, if α, β in k satisfy the equation

$$8(\alpha^5 + \beta^5) = 1,$$

then the plane

$$\begin{cases} x + y &= 2\alpha v \\ z + u &= 2\beta v \end{cases}$$

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$$\begin{array}{rcl} x + y & = & 2\alpha v \\ z + u & = & 2\beta v \\ \sqrt{\alpha}((x-y)^2 + \alpha^2 v^2) & = & \pm\sqrt{-\beta}((z-u)^2 + \beta^2 v^2). \end{array}$$

We can incorporate the square roots in our definitions: if α and β in k satisfy the equation

$$8(\alpha^{10} + \beta^{10}) = 1,$$

then the conic

$$\begin{aligned} x + y &= 2\alpha^2 v \\ z + u &= 2\beta^2 v \\ \alpha((x-y)^2 + \alpha^4 v^2) &= i\beta((z-u)^2 + \beta^4 v^2) \end{aligned}$$

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(We are counting twice every conic, since the simultaneous change in sign in α and β defines the same conic.)

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parameterizes (some of) the conics on X. Homogenizing and extracting roots, the curve becomes

$$X^{10} + Y^{10} + Z^{10} = 0.$$

Theorem

Let k be an algebraically closed field of characteristic not 2 or 5. The space of conics in the Fermat quintic threefold X over k contains 375 curves that are (doubly covered by) the plane Fermat curve of degree ten

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Remark. There are conics on X that are not contained in any one-parameter family.

Curves of higher degree?

Recall that the components of the curve parameterizing the lines on X are isomorphic to the curve

$$X^5 + Y^5 + Z^5 = 0,$$

and we found that the space of conics contains components that are related to the curve

$$X^{5 \cdot 2} + Y^{5 \cdot 2} + Z^{5 \cdot 2} = 0.$$

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Question: is the plane curve

$$X^{5\cdot d} + Y^{5\cdot d} + Z^{5\cdot d} = 0$$

related to the space of rational curves of degree d contained in X?

Curves of higher degree?

Philip Candelas mentioned a description of the space of lines on the Fermat quintic threefold in which the standard toric variety structure $(\mathbb{C}^\times)^2 \subset \mathbb{P}^2(\mathbb{C})$ plays a natural role.
This is very interesting, since the curves above are obtained by raising to the d-th power the coordinates and are therefore the inverse image under the map

$$\begin{array}{rccc} (-)^d \colon & (\mathbb{C}^\times)^2 & \longrightarrow & (\mathbb{C}^\times)^2 \\ & (x,y) & \longmapsto & (x^d, y^d). \end{array}$$

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Special Points on the Fermat quintic threefold X

In 1967, Lander and Parkin observed that the equality

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

holds: the point $LP := [27, 84, 110, 133, -144]$ lies on X.

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There are no lines through LP.

Are there conics contained in X and containing the point LP?

Conics through special points

The one-parameter families found above span the surface contained in X having equations

$$\begin{aligned} x^5 + y^5 + z^5 + u^5 + v &= 0 \\ (x+y)(x^2+y^2)^2 + (z+u)(z^2+u^2)^2 &= 0 \end{aligned}$$

up to permutation and multiplication by fifth roots of unity of the variables.

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As before, we search modulo small primes for conics containing the reduction of the point LP.

Conics through special points

Lower bounds for non-singular conics through LP

prime p	$\#\begin{pmatrix}\text{conics modulo } p \\ \text{through } LP\end{pmatrix}$
2	1 (singular)
3	1
5	—
7	6
11	4
13	2
17	1
19	6
23	1
29	0
31	101

Conics through special points

Lower bounds for non-singular conics through LP

p	# (conics)	lines through LP?
2	1 (sing)	*Yes*
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5	–	–
7	6	*Yes*
11	4	*Yes*
13	2	*Yes*
17	1	*Yes*
19	6	*No*
23	1	*No*
29	0	*No*
31	101	*Yes*

LP not on a line $\pmod{19}$

$\Longrightarrow$ no line over $\mathbb{Q}$

"$\Longleftarrow$" $31 \equiv 1 \pmod 5$

Conics through special points

p	3	7	11	13	17	19	23	29
# (conics)	1	6	4	2	1	6	1	0

(We omit the values for the primes 2, 5 and 31.)

Remember that the number of conics above is only a **lower bound**.

The space parameterizing non-singular conics through LP is also an algebraic variety, denote it by $Con(X, LP)$.

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Thus, not only they represent *non-singular conics*, but they

are also non-singular points of the variety $Con(X, LP)$.

Conclusions from the previous data

The reason for being excited about the *smoothness* of the points in the variety of conics modulo primes is Hensel's Lemma.

Lemma (Hensel's Lemma)

Let $f(x)$ be a polynomial with integer coefficients in one variable and let p be a prime. Suppose that x_0 is an integer such that

$$f(x_0) \equiv 0 \pmod p \quad and \quad f'(x_0) \not\equiv 0 \pmod p.$$

Then there is a p-adic integer $\overline{x}_0 \in \mathbb{Z}_p$ such that $f(\overline{x}_0) = 0$.

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We restate (and mildly generalize) Hensel's Lemma as follows.

Let p be a prime and let Y be an algebraic variety defined and flat over $\mathbb{Z}$. Suppose that y is a point of the reduction of Y modulo p and that it is non-singular, then Y has a p-adic point.

Conclusions from the previous data

p	3	7	11	13	17	19	23	29
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IF the space $Con(X, LP)$ were flat near a pair $(p, \mathrm{conic} \pmod p)$

THEN there would be a conic through LP over $\mathbb{C}$!

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The amazing conclusion is that knowing a *smooth* point modulo a finite field, we would deduce the existence of a point over a field of characteristic zero!

Note that this rests on the hypothesis that we can prove flatness. This is something that we do **not** know how to do at the moment.

Short summary

$X = \{x^5 + y^5 + z^5 + u^5 + v^5 = 0\} \subset \mathbb{P}^4$ Fermat quintic threefold

Lines on X: parameterized by curves isomorphic to
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Over many finite fields, yes. Is this enough to conclude that LP is contained in a conic over $\overline{\mathbb{Q}}$?

Thank you!