# BILINEAR PAIRINGS

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Let $R$ be a domain, $K = \mathrm{Frac}(R)$, $E, F$ finitely generated free $R$ modules of the same rank, say $n$, and $S = R - \{0\}$. Throughout these notes we will work in this setup, even if the definitions that follow can be given in greater generality.

**Definition 1:** An $R$**- bilinear pairing** $H : E \times F \to R$ is an $R$- bilinear map.

Note that for all $e \in E$, we get an $R$- linear map $H_e : F \to R$, where $H_e = H(e, \ \ )$. So, $H_e$ is an element of the dual module $\mathrm{Hom}_R(F, R)$. By bilinearity of $H$, we actually get a homomorphism of $R$ modules $\overline{H} : E \to \mathrm{Hom}_R(F, R)$.

**Definition 2:** A bilinear pairing $H : E \times F \to R$ is **perfect** if $\overline{H} : E \to \mathrm{Hom}_R(F, R)$ is an isomorphism of $R$ modules.

**Definition 3:** A bilinear pairing $H : E \times F \to R$ is **nondegenerate** if $\overline{H} : E \to \mathrm{Hom}_R(F, R)$ is injective.

In particular, a perfect pairing is nondegenerate.

**Definiton 4:** Let $e_1, ..., e_n$, $f_1, ..., f_n$ be bases for $E$, $F$ respectively. Then, given a bilinear pairing $H : E \times F \to R$, the **matrix** of $H$, denoted $M_H$ is defined as $M_H = (H(e_i, f_j))$ for $1 \leq i, j \leq n$.

**Result 5:** If $f_1^*, ..., f_n^*$ denotes the dual basis of $\mathrm{Hom}_R(F, R)$, dual to $f_1, ..., f_n$, then $M_H$ is the matrix of the linear map of free modules $\overline{H} : E \to \mathrm{Hom}_R(F, R)$.

**Proof:** Omitted. $\boxdot$

**Result 6:** Let $S = R - \{0\}$. Then any bilinear pairing $H : E \times F \to R$ gives a bilinear pairing $S^{-1}H : S^{-1}E \times S^{-1}F \to K$ of $K$ vector spaces.

**Proof:** For $(\frac{e}{s}, \frac{f}{t}) \in S^{-1}E \times S^{-1}F$, define $S^{-1}H(\frac{e}{s}, \frac{f}{t}) = \frac{H(e,f)}{st}$. Then it is easily verified that $S^{-1}H$ is a bilinear map. $\boxdot$

Note that $\frac{e_1}{1}, ..., \frac{e_n}{1}$ is a $K$ basis for $S^{-1}E$, and similarly $\frac{f_1}{1}, ..., \frac{f_n}{1}$ is a $K$ basis for $S^{-1}F$. Since $R$ is domain, $R$ can be naturally identified as a subring of $K$. Under this identification, it easily seen that $M_H = M_{S^{-1}H}$. Since $S^{-}H$ is a bilinear pairing of vector spaces, hence a perfect pairing is the same thing as a nondegenerate pairing, because any injective map of finite dimensional vector spaces of the same dimension is surjective.

**Result 7:** If $S^{-1}H$ is perfect/ nondegenerate, then $M_H$ has maximal rank $n$ and $H$ is nondegenerate.

**Proof:** If $S^{-1}H$ is nondegenerate, then $\overline{S^{-1}H} : S^{-1}E \to \mathrm{Hom}_K(S^{-1}F, K)$ is an isomorphism. By Result 5, $M_{S^{-1}H}$ has maximal rank. Since $M_H = M_{S^{-1}H}$, it follows that $M_H$ must have maximal rank over $R$. Note that $\overline{S^{-1}H} : S^{-1}E \to \mathrm{Hom}_K(S^{-1}F, K)$ is just the localization of $\overline{H} : E \to \mathrm{Hom}_R(F, R)$. If the latter map were not injective, then $\overline{S^{-1}H} : S^{-1}E \to \mathrm{Hom}_K(S^{-1}F, K)$ would also not be injective, contradicting the fact that this map is an isomorphism. So, $\overline{H} : E \to \mathrm{Hom}_R(F, R)$ is injective, and $H$ is nondegenerate. $\boxdot$

**Result 8:** If $H$ is nondegenerate, then $S^{-1}H$ is perfect/ nondegenerate.

**Proof:** $H$ nondegenerate $\Rightarrow$ the $R$-linear map $\overline{H} : E \to \mathrm{Hom}_R(F, R)$ is injective $\Rightarrow$ $\overline{S^{-1}H} : S^{-1}E \to \mathrm{Hom}_K(S^{-1}F, K)$ is injective (by exactness of localization), hence surjective, because $S^{-1}E, \mathrm{Hom}_K(S^{-1}F, K)$ have the same dimension over $K$. $\boxdot$

So, we obtain the following result:

**$H$ is nondegenerate if and only if $S^{-1}H$ is perfect/ nondegenerate.**