EXERCISE 11

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Exercise 11: Show that every conic is either a double line, a union of lines, or has the property that it meets every line in $0, 1$ or 2 points.

Proof: Let k be a field and $F \in k[X_0, X_1, X_2]$ define a conic $F = 0$, where $F = \sum_{1 \leq i \leq j \leq 2} a_{ij} X_i X_j$, $a_{ij} \in k$, and a_{ij} are not all 0. Either F is irreducible or it isn't. If F is not irreducible, then $\exists$ homogeneous non-constant $g, h \in k[X_0, X_1, X_2]$ such that $F = gh$ (It is easy to see why g, h have to be homogeneous). Since F has degree 2, it follows that g, h must both have degree 1. So, they must be of the form $g = aX_0 + bX_1 + cX_2$ and $h = dX_0 + eX_1 + fX_2$. Now two things can happen:

(1) For all $\lambda \in k$, $g \neq h$, in which case $F = 0$ is actually two lines.

(2) $\exists\ \lambda \in k^*$ such that $h = \lambda g$, in which case $F = 0$ is a double line.

Let us now deal with the case where F is irreducible. We want to show that in this case $F = 0$ intersects a line in $\mathbb{P}^2$ in 0, 1 or 2 points. A line in $\mathbb{P}^2$ is given in its parametric form by $[at + bs : ct + ds : s]$ where t, s are the parameters that range over k such that they are not both simultaneously 0 (otherwise they will not define a point in $\mathbb{P}^2$), and a, c are not both 0. We will try to analyze what happens when our parametric line intersects the conic $F = 0$ in the affine part, i.e., when $s \neq 0$ and so can be taken to be equal to 1, and when $s = 0$.

When $s = 1$, then the points on the line are given by $[at+b : ct+d : 1]$ as t ranges over the elements of k. Note that any such point is on the conic if and only if $F(at+b, ct+d, 1) = 0$. So, to see which points on the affine part of our parametric line is on the irreducible conic $F = 0$, it suffices to solve the equation $F(at + b, ct + d, 1) = 0$ where the LHS of the equation is a polynomial of degree at most 2 in t. Since $F(at + b, ct + d, 1)$ is at most of degree 2, we have that the affine part of our line intersects the conic $F = 0$ in 0, 1 or 2 points.

If the affine part of the line intersects the conic $F = 0$ at 0 points, then it suffices to show that there are at most 2 points on the parametric line with $s = 0$ which intersects our conic $F = 0$. But, this is clear, because by an argument similar to the one given in the last paragraph, any such point has to be a solution of $F(at, ct, 0) = 0$, where the LHS is again a polynomial of degree at most 2 in t.

If the affine part of the line intersects the conic $F = 0$ at 2 points (where we count the points with their multiplicity), then $F(at + b, ct + d, 1)$ has to be a quadratic in t. The leading coefficient of $F(at + b, ct + d, 1)$ is $a_{00}a^2 + a_{11}c^2 + a_{01}ac$, which must be non-zero. It suffices to show that no point on the line with $s = 0$ intersects our conic $F = 0$. Now $F(at, ct, 0) = (a_{00}a^2 + a_{11}c^2 + a_{01}ac)t^2$. Since $a_{00}a^2 + a_{11}c^2 + a_{01}ac \neq 0$, it follows that $F(at, ct, 0) = 0$ if and only if $t = 0$. But, it $t = 0$, then we do not get a point in $\mathbb{P}^2$. So, there is no point on the line with $s = 0$ which intersects our conic. Hence, in this case the conic and our line intersect at exactly 2 points (1 point if we disregard the multiplicity of a double root).

We are left with the case when the affine part of our line intersects the conic at 1 point. Well, in this case $F(at+b, ct+d, 1)$ must be a linear polynomial in t. So, the coefficient of the t^2 term, which is $a_{00}a^2 + a_{11}c^2 + a_{01}ac$ must be 0. But, if this is indeed the case, then it means that $[at : ct : 0]$, as t ranges over k^*, are points both on the conic and the line. But, we are working on the projective plane. So, if $t \neq 0$, then $[at : ct : 0] = [a : c : 0]$. So, we again get exactly two points of intersection of the conic and the line, namely the point in the affine part, and the point $[a : c : 0]$. This completes the proof.