EXERCISE 28

Suppose that M is a finitely generated and graded R-module for some ring R. Letting $\{a_1, \cdots, a_\ell\} \subset M$ be the generating set, we can write any $x \in M$ as

$$x = r_1 a_1 + \cdots + r_\ell a_\ell$$

for some $r_1, \cdots r_\ell \in R$ by definition. Since M is graded, this implies that

$$x = r_1 \sum_{n \in \mathbb{Z}} m_{1,n} + \cdots + r_\ell \sum_{n \in \mathbb{Z}} m_{\ell,n},$$

where the $m_{i,n} \in M_n$ and each summation term contains finitely many nonzero $m_{i,n}$ by definition of direct sum. Letting k be the maximum number of nonzero elements for the ℓ summation terms, we can rewrite the above as

$$x = r_1(m_{1,1} + \cdots + m_{1,k}) + \cdots + r_\ell(m_{\ell,1} + \cdots + m_{\ell,k}),$$

where each $m_{i,j}$ is in some M_n. This shows that the finite set of homogenous elements $\{m_{i,j}\}$ where $1 \leq i \leq \ell$ and $1 \leq j \leq k$ generates M. ■