EXERCISE 31

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Exercise 31: Which pairs a, b can occur in the lemma? In other words: What are all possible Hilbert polynomials of finitely generated graded R-modules, where $R = k[S, T]$?

Solution: The lemma shows that given a finitely generated module M over R, the Hilbert function $H_M : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ is polynomial like, and moreover, $\exists\ a, b \in \mathbb{Z}$ such that $H_M(n) = an + b$ for all $n >> 0$. Since $H_M(n)$ is nonnegative, the first thing one can say is that $a \geq 0$.

I will first show that there is a module M such that the Hilbert function of M is given by $an + b$, for any $a, b \in \mathbb{Z}$ such that $a > 0$.

Well, just let $M = \underbrace{R \oplus R \oplus ... \oplus R}_{(a-1) times} \oplus R(b - a)$. Then it is easily seen that the Hilbert polynomial of this module is $an + b$.

If $a = 0$, then again by the fact that H_M is nonnegative, it is clear that $b \geq 0$. We get $b = 0$ for the zero module. If $b > 0$, then since $a = 0$, we get that H_M is constant for $n >> 0$. Now, $R/(T)$ is an R algebra, hence an R module, where multiplication by T acts as 0 on this module. So, $R/(T)$ is actually a module over $k[S]$, and it is clear that the Hilbert polynomial of this module is just the constant function 1. Hence, $M = \underbrace{R/(T) \oplus ... \oplus R/(T)}_{b-times}$ is a module whose Hilbert polynomial is just the constant b.

In conclusion, (a, b) takes values in the set $\{(a, b) | a, b \in \mathbb{Z}, a > 0\} \cup \{(0, b) : b \in \mathbb{Z}_{\geq 0}\}$.