EXERCISE 32

Given a graded free module M over $R = K[S,T]$ for a field K and the values of its Hilbert function H_M, we want to produce an algorithm that finds the splitting type of M.

Let $r, i = 0$, and take $n << 0$. As we've previously seen, taking $n << 0$ guarantees that $H_M(n) = 0$. Now iteratively let $r = r_{old} + H_M(n) - i_{old}$, $i = i_{old} + r_{old} + 2(H_M(n) - i_{old})$, and $n = n_{old} + 1$. In each iteration, add $H_M(n) - i_{old}$ elements equal to $-n$ to the collection of e_i, reordering the collection as needed.

As an example, consider what we did in class, where we began with $n = -3$:
$H_M(-3) = 0$, so $r = 0 + 0 - 0 = 0$ and $i = 0 + 0 + 0 = 0$.
$H_M(-2) = 1$, so $r = 0 + 1 - 0 = 1$ and $i = 0 + 0 + 2 = 2$ (Add 2 to $\{e_i\}$).
$H_M(-1) = 3$, so $r = 1 + 3 - 2 = 2$ and $i = 2 + 1 + 2 = 5$ (Add 1 to $\{e_i\}$).
$H_M(0) = 5$, so $r = 2 + 5 - 5 = 2$ and $i = 5 + 2 + 0 = 7$.
$H_M(1) = 7$, so $r = 2 + 7 - 7 = 2$ and $i = 7 + 2 + 0 = 9$.
$H_M(2) = 9$, so $r = 2 + 9 - 9 = 2$ and $i = 9 + 2 + 0 = 11$.
$\cdots$

Note that without more information, it is impossible to decide when this process should terminate. However, If we know the *rank* of M, we know to stop as soon as $r = rank(M)$. Since the *Hilbert polynomial* of M takes the form $rn + e_1 + \cdots + e_r + r$ where $r = rank(M)$, knowing its explicit form $an + b$ is enough to deduce that $rank(M) = a$ so this can also be useful. $\blacksquare$.