Exercises

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1 The Projective Plane

1.1 Exercise 2.

Proposition. *Let k be a field. The projective plane $\mathbb{P}^2_k$ defined as the space of lines in k^3 is an axiomatic projective plane.*

Proof. Let $\pi : k^3 \to \mathbb{P}^2_k$ be the quotient map. The preimage of a point under π is a line in k^3. Two distinct points in $\mathbb{P}^2$ determine two distinct lines in k^3 whose span is a hyperplane, which projects to a unique line in $\mathbb{P}^2$. Similarly, two distinct lines in $\mathbb{P}^2$ lift to a pair of hyperplanes in k^3 which intersect in a unique line, which projects to a unique point. Thus $\mathbb{P}^2_k$ satisfies the axioms of a projective plane. ■

1.2 Exercise 3.

Proposition. *Let $k = \mathbb{F}_q$. $\mathbb{P}^2_k$ contains $q^2 + q + 1$ points.*

Proof. We enumerate the points $(x : y : z) \in \mathbb{P}^2_k$ in cases. First, suppose $x \neq 0$. Then fix $x = 1$, so the coordinates y and z may vary freely in k, yielding q^2 points. Now suppose $x = 0$ and $y \neq 0$. Fix $y = 1$ and let z vary over k, yielding q points. Finally, suppose $x, y = 0$, so $z \neq 0$ by definition of $\mathbb{P}^2$. Then there is only one point under these conditions, so the total number of points is $q^2 + q + 1$. ■

Remark. In general, $\mathbb{P}^n$ decomposes into pieces given by whether or not the first coordinate is zero, which are isomorphic to k^n and $\mathbb{P}^{n-1}$. Repeated application of this decomposition gives $\mathbb{P}^n = k^n \cup k^{n-1} \cup ... \cup k^2 \cup k \cup \{\infty\}$, which has cardinality

$$\sum_{i=0}^{n} q^i.$$

Alternately, we can consider the effect of the group action, which partitions $k^{n+1} \setminus \{0\}$ into orbits, each of which has cardinality $q - 1$. Since $|k^{n+1} \setminus \{0\}\,| = q^{n+1} - 1$, the quotient under the action has $(q^{n+1} - 1)/(q - 1)$ points; this value is of course equal to the enumerative calculation.

2 The Projective Line

2.1 Exercise 7.

The general case is done in the section on the projective plane.

3 Conics in $\mathbb{P}^2$

3.1 Exercise 10.

The conic $C : x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ has no points since the terms are positive definite and $(0 : 0 : 0) \notin \mathbb{P}^2_{\mathbb{R}}$.

4 Morphisms of Projective Space

4.1 Exercise 17a.

Proposition. *Any degree two morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ maps onto either a line or a conic.*

Beginning of Proof. Such a morphism is given by a relatively prime triple G_0, G_1, G_2 of homogeneous quadratics in $k[Y_0, Y_1]$. If the morphism is into a line or conic, then it is certainly onto since the G_i are nonconstant. To show that the morphism is into, we must find a line $L : F = 0$ or a conic $C : H = 0$ such that $F(G_0, G_1, G_2) = 0$ or $H(G_0, G_1, G_2) = 0$; in fact it suffices to find a conic, since any line squares to a reducible conic. We need $a_{ij} \in K$ not all zero such that

$$\sum_{i \leq j} a_{ij} G_i G_j \equiv 0.$$

The terms are elements of a five-dimensional vector space of homogeneous polynomials of degree 4, which has basis $S^4, S^3T, ..., T^4$. There are six polynomials in the sum, hence there is a nontrivial linear dependence. ■

5 Cubics in $\mathbb{P}^2$

5.1 Exercise 18.

We can trivially get $C(K) \subset C'(K)$ if $C(K)$ is empty, so recall Exercise 10, in which we saw that for $C : x^2+y^2+z^2 = 0$, $C(\mathbb{R})$ is empty. If $C' : x^2-y^2+z^2$, then clearly $C(\mathbb{R}) = \emptyset \subset C'(\mathbb{R})$, but $C \not\subset C'$ since $x^2 - y^2 + z^2 \notin (x^2 + y^2 + z^2) \subset \mathbb{R}[x, y, z]$.

5.2 Exercise 20.

I've done a fair bit of random calculation trying to obtain such an intersection over a finite field, but it's hard enough getting four points on a conic at all. Over the complex numbers it should not be a problem; two conics intersecting transversely (so with multiplicity 1 at each intersection point, if I recall correctly) over an algebraically closed field have four intersection points. Let $C : F = 0$ and $C' : F' = 0$, where $F = 2x^2 + y^2 + z^2$ and $F' = x^2 + 2y^2 + 3z^2$. These most likely work, although it is too late to do intersection calculations.