EXERCISE 11

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Exercise 11: Show that every conic is either a double line, a union of lines, or has the property that it meets every line in 0, 1 or 2 points.

Proof: Let $k$ be a field and $F \in k[X_0, X_1, X_2]$ define a conic $F = 0$, where $F = \sum_{1 \leq i \leq j \leq 2} a_{ij} X_i X_j$, $a_{ij} \in k$, and $a_{ij}$ are not all 0. Either $F$ is irreducible or it isn’t. If $F$ is not irreducible, then $\exists$ homogeneous non-constant $g, h \in k[X_0, X_1, X_2]$ such that $F = gh$ (It is easy to see why $g, h$ have to be homogeneous). Since $F$ has degree 2, it follows that $g, h$ must both have degree 1. So, they must be of the form $g = aX_0 + bX_1 + cX_2$ and $h = dX_0 + eX_1 + fX_2$.

Now two things can happen:

1) For all $\lambda \in k$, $g \neq h$, in which case $F = 0$ is actually two lines.

2) $\exists \lambda \in k^*$ such that $h = \lambda g$, in which case $F = 0$ is a double line.

Let us now deal with the case where $F$ is irreducible. We want to show that in this case $F = 0$ intersects a line in $P^2$ in 0, 1 or 2 points. A line in $P^2$ is given in its parametric form by $[at + bs : ct + ds : s]$ where $t, s$ are the parameters that range over $k$ such that they are not both simultaneously 0 (otherwise they will not define a point in $P^2$), and $a, c$ are not both 0. We will try to analyze what happens when our parametric line intersects the conic $F = 0$ in the affine part, i.e., when $s \neq 0$ and so can be taken to be equal to 1, and when $s = 0$.

When $s = 1$, then the points on the line are given by $[at + b : ct + d : 1]$ as $t$ ranges over the elements of $k$. Note that any such point is on the conic if and only if $F(at + b, ct + d, 1) = 0$. So, to see which points on the affine part of our parametric line is on the irreducible conic $F = 0$, it suffices to solve the equation $F(at + b, ct + d, 1) = 0$ where the LHS of the equation is a polynomial of degree at most 2 in $t$. Since $F(at + b, ct + d, 1)$ is at most of degree 2, we have that the affine part of our line intersects the conic $F = 0$ in 0, 1 or 2 points.

If the affine part of the line intersects the conic $F = 0$ at 0 points, then it suffices to show that there are at most 2 points on the parametric line with $s = 0$ which intersects our conic $F = 0$. But, this is clear, because by an argument similar to the one given in the last paragraph, any such point has to be a solution of $F(at, ct, 0) = 0$, where the LHS is again a polynomial of degree at most 2 in $t$. 

If the affine part of the line intersects the conic \( F = 0 \) at 2 points (where we count the points with their multiplicity), then \( F(at + b, ct + d, 1) \) has to be a quadratic in \( t \). The leading coefficient of \( F(at + b, ct + d, 1) \) is \( a_{00}a^2 + a_{11}c^2 + a_{01}ac \), which must be non-zero. It suffices to show that no point on the line with \( s = 0 \) intersects our conic \( F = 0 \). Now \( F(at, ct, 0) = (a_{00}a^2 + a_{11}c^2 + a_{01}ac) t^2 \). Since \( a_{00}a^2 + a_{11}c^2 + a_{01}ac \neq 0 \), it follows that \( F(at, ct, 0) = 0 \) if and only if \( t = 0 \). But, if \( t = 0 \), then we do not get a point in \( \mathbb{P}^2 \). So, there is no point on the line with \( s = 0 \) which intersects our conic. Hence, in this case the conic and our line intersect at exactly 2 points (1 point if we disregard the multiplicity of a double root).

We are left with the case when the affine part of our line intersects the conic at 1 point. Well, in this case \( F(at + b, ct + d, 1) \) must be a linear polynomial in \( t \). So, the coefficient of the \( t^2 \) term, which is \( a_{00}a^2 + a_{11}c^2 + a_{01}ac \) must be 0. But, if this is indeed the case, then it means that \([at : ct : 0]\), as \( t \) ranges over \( k^* \), are points both on the conic and the line. But, we are working on the projective plane. So, if \( t \neq 0 \), then \([at : ct : 0] = [a : c : 0]\). So, we again get exactly two points of intersection of the conic and the line, namely the point in the affine part, and the point \([a : c : 0]\). This completes the proof.