EXERCISE 29

RANKEYA DATTA

Exercise 29: Let $R = k[t]$ for a field $k$ considered as a graded ring with usual grading. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded $R$ module. If $M$ is finitely generated as an $R$ module, then

(a) $M_n = 0$ for all $n < 0$.

(b) For all $n \in \mathbb{Z}$, $\dim_k(M_n) < \infty$.

Proof: (a) By Exercise 28 it follows that if $M$ is finitely generated, then it can be generated as an $R$ module by a finite set of homogeneous elements. Let $\{m_1, ..., m_t\}$ be a finite set of homogeneous generators for $M$. Let $d$ be the least degree of the generators of this set. Since multiplication by an element of $R$ just raises the degree, it follows that for all $n < d$, $M_n = 0$.

(b) Let $n \in \mathbb{Z}$. Let $\Gamma = \{m_{i_1}, ..., m_{i_s}\} \subset \{m_1, ..., m_t\}$ be the subset of the generating set, consisting of elements of degree less than or equal to $n$. Note that $\Gamma$ could be the empty set, in which case $M_n = 0$, and hence trivially finite dimensional over $k$. Suppose $m_{i_j}$ has degree $d_j$ for all $j \in \{1, ..., s\}$. Then it is easy to see that $M_n$ is generated as a $k$-vector space by $\{t^{d_1}m_{i_1}, t^{d_2}m_{i_2}, ..., t^{d_s}m_{i_s}\}$. Thus, $\dim_k(M_n) < \infty$. This method can be generalized to the case where $R = k[t_1, ..., t_m]$. 