Projective Planes

Exercise 1: Prove that the axiomatic projective plane has the same number of points as lines.

Pf: Apart from the 2 axioms of the projective plane mentioned in the notes, we assume the following additional axioms -
   (3) A projective plane has at least 3 non-collinear points
   (4) Any line in the projective plane passes through at least 3 distinct points.

We will denote our projective plane by $\mathbb{P}$ and define
\[ \mathcal{L} := \{ \text{lines in } \mathbb{P} \} \]
\[ \mathcal{T} := \{ \text{points in } \mathbb{P} \} \]

We divide the proof into 2 cases:

Case 1: $\mathcal{L}$, $\mathcal{T}$ are both infinite sets

Pf of Case 1: Let $\Delta_{\mathcal{L}}$, $\Delta_{\mathcal{T}}$ denote the diagonal of $\mathcal{L} \times \mathcal{L}$ and $\mathcal{T} \times \mathcal{T}$ respectively.
We need to show that $\mathcal{L}$ and $\mathcal{T}$ have the same cardinality.
It is easy to see that
\[ |\mathcal{L}| = |\mathcal{L} \times \mathcal{L}| = |\mathcal{L} \times \mathcal{L} - \Delta_{\mathcal{L}}| \]
\[ |\mathcal{T}| = |\mathcal{T} \times \mathcal{T}| = |\mathcal{T} \times \mathcal{T} - \Delta_{\mathcal{T}}| \]

By axioms (1), (2) of the axiomatic projective plane we have natural maps
\[ \Pi_1 : \mathcal{L} \times \mathcal{L} - \Delta \mathcal{L} \rightarrow \mathcal{T} \\
(\ell_1, \ell_2) \mapsto \ell_1 \cap \ell_2 \]

\[ \Pi_2 : \mathcal{T} \times \mathcal{T} - \Delta \mathcal{T} \rightarrow \mathcal{L} \\
(p, q) \mapsto \overline{pq} \]

where \( \overline{pq} \) denotes the unique line through \( p \) and \( q \).

Let us show that \( \Pi_1, \Pi_2 \) are surjective.

If \( p \in \mathcal{T} \), then by axiom (3), and the fact \( \exists \) distinct points \( q \) and \( r \), such that \( p, q, r \) are not collinear. Then clearly \( \overline{pq} \neq \overline{pr} \) and \( \Pi_1(\overline{pq}, \overline{pr}) = p \). This shows that \( \Pi_1 \) is surjective.

Let \( l \in \mathcal{L} \). Then by axiom (4), \( l \) has at least 2 distinct points \( p, q \) on it. Again, clearly \( \Pi_2(p, q) = \overline{pq} = l \). So, \( \Pi_2 \) is surjective.

\( \Pi_1 \) surjective \( \Rightarrow \) \( |\mathcal{T}| \leq |\mathcal{L} \times \mathcal{L} - \Delta \mathcal{L}| = |\mathcal{L}| \)

\( \Pi_2 \) surjective \( \Rightarrow \) \( |\mathcal{L}| \leq |\mathcal{T} \times \mathcal{T} - \Delta \mathcal{T}| = |\mathcal{T}| \)

Thus, \( |\mathcal{T}| \leq |\mathcal{L}| \leq |\mathcal{T}| \Rightarrow |\mathcal{L}| = |\mathcal{T}| \)

Note that if \( \mathcal{L} \) is infinite then by axioms (1) and (4), \( \mathcal{T} \) must be infinite and we reduce to case 1. (Here axiom (4) is used in the sense that it guarantees that every line has a point on it.)

If \( \mathcal{T} \) is infinite, suppose that \( \mathcal{L} \) is finite. By axioms (2) and (1), every point lies on some line. So, \( \exists l \in \mathcal{L} \) such that \( l \) has infinitely many points on it. But, by axiom (3) again,
∃ p ∈ T such that p Φ l. But then for any q ∈ T such that q Φ l, we have a line pq which is distinct from l, and by axiom (2), if q, q' ∈ T such that q ≡ q' and q, q' Φ l, then pq ≡ pq'. So, this gives us infinitely many distinct lines through p intersecting l. Thus, L is infinite, a contradiction. So, L must have been infinite to begin with, and we again reduce to Case 1.

Case 2: L, T are both finite sets.

We will do this proof in points.

Claim 1: Let p ∈ T. If Lp denotes the set of all lines passing through p, then #Lp is independent of our choice of p.

iff of claim 1: Let p, q ∈ T be two distinct points. It suffices to show that #Lp = #Lq.

By axiom (1), ∃! line pq passing through p and q.

Now by axiom (4), ∃ a point r on pq distinct from p and q.

Let l ∈ Lp - {pq}, m ∈ Lq - {pq}. By axiom (2), l and m are distinct. By axiom (2), l ∩ m is a single point which
is clearly not on $pq$. Let $\Gamma_{p,q}$ denote the set of points of $Tp$ not on $pq$. Then we get a map

$$\varphi: (L_p - \{p\}) \times (L_q - \{q\}) \to \Gamma_{p,q}$$

$$(l, m) \mapsto l \cap m$$

Note that $\Gamma_{p,q} \neq \emptyset$ by axiom (3). The map $\varphi$ is a bijection with inverse

$$\tau: \Gamma_{p,q} \to (L_p - \{p\}) \times (L_q - \{q\})$$

$s \mapsto (\widetilde{ps}, \tilde{ts})$

Thus, $(\# L_p - 1) (\# L_q - 1) = \# \Gamma_{p,q}$. One can similarly show that

$$(\# L_q - 1) (\# L_p - 1) = \# \Gamma_{p,q}.$$ 

Thus, $\# L_p - 1 = \# L_q - 1 \Rightarrow \# L_p = \# L_q$.

This proves claim 1.

Let us denote the number of lines through any point, which is a constant, by $c$.

Claim 2: Let $l \in L$. Let $T_d$ denote the set of points on $Tl$ passing through $l$. Then $\# T_d$ is irrespective of $l$, and $\# T_d = c$ for all $l \in L$.

Proof of claim 2: Let $p$ be a point not on $L$. Again such a $p$ exists by axiom 3.
In particular $l \not\in \mathcal{L}_p$. Define a map

$$
\chi : \mathcal{L}_p \rightarrow T_d
$$

$$
\eta \mapsto l \cap \eta
$$

Then $\chi$ has inverse

$$
\chi^{-1} : T_d \rightarrow \mathcal{L}_p
$$

$$
s \mapsto \overline{ps}
$$

So, $\# T_d = \# \mathcal{L}_p = c$ by claim 1. Since $l$ was arbitrary, this proves claim 2.

We are now in a position to prove Case 2. We will basically count the number of points and the number of lines and show that these two numbers agree.

Let $p, q \in T$ be 2 distinct points. Note that

$$
T = T_{p - \overline{p}} \cup T_{q - \overline{q}} \text{ where } T_{p - \overline{p}} \cap T_{q - \overline{q}} = \emptyset.
$$

Now, by claim 1, $\# T_{p - \overline{p}} = (c-1)(c-1)$ and by claim 2, $\# T_{\overline{p}} = c$. So,

$$
\# T = (c-1)(c-1) + c = c^2 - 2c + 1 + c = c(c-1) + 1.
$$

On the other hand, let $l \in \mathcal{L}$.

Then

$$
\mathcal{L} = \bigsqcup_{q \in T_d} (\mathcal{L}_q - \{l\}) \sqcup \{l\} \quad \text{(I use the disjoint union symbol just to emphasize that the sets are mutually disjoint)}
$$
Now, by claim 2, $\# T_x = c$ and by claim 1, $\# \mathcal{L}_x - \{x\} = c - 1$. Thus, $\# \mathcal{L} = c(c-1) + 1$. \qed