WHY DEGREE 5 MORPHISMS ARE NOT FREE

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We will work throughout over $k = \mathbb{F}_2$. Let $R = k[S, T]$. Any homogeneous polynomial of degree 5 in $R$ is of the form $aS^5 + bS^4T + cS^3T^2 + dS^2T^3 + eST^4 + fT^5$. Raising this polynomial to the 5th power we get

The coefficient of the $S^{23}T^2$ term = $a^4c$
The coefficient of the $S^{19}T^6$ term = $b^4c$
The coefficient of the $S^{15}T^{10}$ term = $c^5$
The coefficient of the $S^{11}T^{14}$ term = $d^4c$
The coefficient of the $S^7T^{18}$ term = $e^4c$
The coefficient of the $S^3T^{22}$ term = $f^4c$.

Now suppose we have a morphism $\varphi = (G_0, ..., G_5)$ that is free, where

$G_0 = a_1S^5 + b_1S^4T + c_1S^3T^2 + d_1S^2T^3 + e_1ST^4 + f_1T^5$
$G_1 = a_2S^5 + b_2S^4T + c_2S^3T^2 + d_2S^2T^3 + e_2ST^4 + f_2T^5$
$G_2 = a_3S^5 + b_3S^4T + c_3S^3T^2 + d_3S^2T^3 + e_3ST^4 + f_3T^5$
$G_3 = a_4S^5 + b_4S^4T + c_4S^3T^2 + d_4S^2T^3 + e_4ST^4 + f_4T^5$
$G_4 = a_5S^5 + b_5S^4T + c_5S^3T^2 + d_5S^2T^3 + e_5ST^4 + f_5T^5$
$G_5 = a_6S^5 + b_6S^4T + c_6S^3T^2 + d_6S^2T^3 + e_6ST^4 + f_6T^5$

Then we know that the matrix

$$\begin{bmatrix}
  a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\
  a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\
  a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\
  a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\
  a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\
  a_6 & b_6 & c_6 & d_6 & e_6 & f_6
\end{bmatrix}$$

is invertible.

I claim then that the matrix
It suffices to show that the rows of this matrix are linearly independent, considered as elements of \( k^6 \). So, suppose we have \( l_1(a_1^4, b_1^4, c_1^4, d_1^4, e_1^4, f_1) + \cdots + l_6(a_6^4, b_6^4, c_6^4, d_6^4, e_6^4, f_6) \), where \( l_1, \ldots, l_6 \in k \). Since \( k \) is algebraically closed, there exists \( m_1, \ldots, m_6 \in k \) such that \( m_1^4 = l_1 \). Then, \( m_1^4(a_1^4, b_1^4, c_1^4, d_1^4, e_1^4, f_1) + \cdots + m_6^4(a_6^4, b_6^4, c_6^4, d_6^4, e_6^4, f_6) = 0 \). So, we have

\[
\begin{align*}
\sum_i m_i^4 a_i^4 &= (\sum_i m_i a_i)^4 = 0 \Rightarrow \sum_i m_i a_i = 0 \\
\sum_i m_i^4 b_i^4 &= (\sum_i m_i b_i)^4 = 0 \Rightarrow \sum_i m_i b_i = 0 \\
\sum_i m_i^4 c_i^4 &= (\sum_i m_i c_i)^4 = 0 \Rightarrow \sum_i m_i c_i = 0 \\
\sum_i m_i^4 d_i^4 &= (\sum_i m_i d_i)^4 = 0 \Rightarrow \sum_i m_i d_i = 0 \\
\sum_i m_i^4 e_i^4 &= (\sum_i m_i e_i)^4 = 0 \Rightarrow \sum_i m_i e_i = 0 \\
\sum_i m_i^4 f_i^4 &= (\sum_i m_i f_i)^4 = 0 \Rightarrow \sum_i m_i f_i = 0
\end{align*}
\]

Since the matrix

\[
\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\
a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\
a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\
a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\
a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\
a_6 & b_6 & c_6 & d_6 & e_6 & f_6
\end{bmatrix}
\]

is invertible, we must have \( m_1 = \cdots = m_6 = 0 \). Hence, \( l_1 = \cdots = l_6 = 0 \), and we are done.

Now, since \( G_3^5 + \cdots + G_5^5 = 0 \), we must have
\[
\begin{align*}
a_1^4 c_1 + a_2^4 c_2 + \cdots + a_6^4 c_6 &= 0 \text{ (coeff. of the } S^{23} T^2 \text{ term)} \\
b_1^4 c_1 + b_2^4 c_2 + \cdots + b_6^4 c_6 &= 0 \text{ (coeff. of the } S^{19} T^6 \text{ term)} \\
c_1^4 c_1 + c_2^4 c_2 + \cdots + c_6^4 c_6 &= 0 \text{ (coeff. of the } S^{15} T^{10} \text{ term)} \\
d_1^4 c_1 + d_2^4 c_2 + \cdots + d_6^4 c_6 &= 0 \text{ (coeff. of the } S^{11} T^{14} \text{ term)} \\
e_1^4 c_1 + e_2^4 c_2 + \cdots + e_6^4 c_6 &= 0 \text{ (coeff. of the } S^{7} T^{18} \text{ term)} \\
f_1^4 c_1 + f_2^4 c_2 + \cdots + f_6^4 c_6 &= 0 \text{ (coeff. of the } S^{3} T^{22} \text{ term)}
\end{align*}
\]
We know that the dot product $\langle \cdot, \cdot \rangle : k^6 \times k^6 \rightarrow k$ is nondegenerate. Since,

$$\begin{bmatrix}
a_1^4 & b_1^4 & c_1^4 & d_1^4 & e_1^4 & f_1^4 \\
a_2^4 & b_2^4 & c_2^4 & d_2^4 & e_2^4 & f_2^4 \\
a_3^4 & b_3^4 & c_3^4 & d_3^4 & e_3^4 & f_3^4 \\
a_4^4 & b_4^4 & c_4^4 & d_4^4 & e_4^4 & f_4^4 \\
a_5^4 & b_5^4 & c_5^4 & d_5^4 & e_5^4 & f_5^4 \\
a_6^4 & b_6^4 & c_6^4 & d_6^4 & e_6^4 & f_6^4
\end{bmatrix}$$

is invertible, its columns must be linearly independent, considered as elements of $k^6$. Hence, $\{(a_1^4, \ldots, a_6^4), (b_1^4, \ldots, b_6^4), (c_1^4, \ldots, c_6^4), (d_1^4, \ldots, d_6^4), (e_1^4, \ldots, e_6^4), (f_1^4, \ldots, f_6^4)\}$ is a basis for $k^6$. But, then we see that $\langle (c_1^4, \ldots, c_6^4), (a_1^4, \ldots, a_6^4) \rangle = \langle (c_1^4, \ldots, c_6^4), (b_1^4, \ldots, b_6^4) \rangle = \langle (c_1^4, \ldots, c_6^4), (c_1^4, \ldots, c_6^4) \rangle = \langle (c_1^4, \ldots, c_6^4), (d_1^4, \ldots, d_6^4) \rangle = \langle (c_1^4, \ldots, c_6^4), (e_1^4, \ldots, e_6^4) \rangle = \langle (c_1^4, \ldots, c_6^4), (f_1^4, \ldots, f_6^4) \rangle = 0$.

So by the nondegeneracy of $\langle \cdot, \cdot \rangle$ it follows that $(c_1, \ldots, c_6) = (0, \ldots, 0)$. But this is impossible since

$$\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\
a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\
a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\
a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\
a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\
a_6 & b_6 & c_6 & d_6 & e_6 & f_6
\end{bmatrix}$$

is invertible. The contradiction proves no free degree 5 morphism can exist.