WEIL $q$-NUMBERS OF DEGREE 4 OVER $\mathbb{Q}$

Let $\pi \in \mathbb{C}$ be a Weil $q$-number of degree 4 over $\mathbb{Q}$. We argue that:

1. For no embedding $\sigma : \mathbb{Q}(\pi) \rightarrow \mathbb{C}$ is the image $\sigma(\pi)$ real. This is true because we saw in the talk that in this case $\pi$ has degree 1 or 2 over $\mathbb{Q}$.

2. Let $P(X) = X^4 + aX^3 + bX^2 + cX + d$ be the minimal polynomial over $\mathbb{Q}$. Note that we can write

   $P(X) = X^4 + aX^3 + bX^2 + cX + d = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)$

   where $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ are all 4 conjugates of $\pi$ in $\mathbb{C}$.

3. In particular, for each $i$ we have $\overline{\alpha_i} \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (where $z \mapsto \overline{z}$ is complex conjugation). Since none of the $\alpha_i$ are real we may choose the numbering such that $\overline{\alpha_1} = \alpha_2$ and $\overline{\alpha_3} = \alpha_4$.

4. Note that $a, b, c, d \in \mathbb{Z}$ because $\pi$ is an algebraic integer.

5. Because $|\alpha_i| = \sqrt{q}$ we see that $|\overline{\alpha_i}| = q/\alpha_i$. In particular we see that the roots of the polynomial $X^4P(q/X)$ are the same as the roots of $P(X)$. By looking at the leading coefficient we deduce that

   $X^4P(q/X) = q^2P(X)$.

Writing this out we obtain

$q^4 +aq^3X + bq^2X^2 + cqX^3 + dX^4 = q^2X^4 + aq^2X^3 + bq^2X^2 + cq^2X + dq^2$.

We conclude that $d = q^2$ and that $c = aq$. Thus we conclude that

$P(X) = X^4 + aX^3 + bX^2 + aqX + q^2$.

6. Note that $a = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$. In particular we have

   $|a| \leq 4\sqrt{q}$

7. Note that $b = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$. In particular we have

   $|b| \leq 6q$

8. Let $\beta = \pi + \overline{\pi} = \pi + q/\pi$. What is the minimal polynomial of $\beta$ over $\mathbb{Q}$? To see this note that $\pi$ is a root of $P(X)/X^2$. We write this as

   $P(X)/X^2 = X^2 + aX + b + aq/X + q^2/X^2$

   $= (X + q/X)^2 - 2q + a(X + q/X) + b$

   $= (X + q/X)^2 + a(X + q/X) + b - 2q$

   $= Y^2 + aY + (b - 2q)$.

where $Y = X + q/X$. In other words, $\beta$ satisfies the equation $Y^2 + aY + (b - 2q) = 0$.

9. In order for $\pi$ to be a Weil $q$-number we know by general theory that $\beta$ has to be totally real. In other words the discriminant $\Delta$ of the quadratic polynomial $Y^2 + aY + (b - 2q)$ has to be positive. We compute $\Delta = a^2 - 4(b - 2q) = a^2 + 8q - 4b$. The resulting inequality is $a^2 + 8q > 4b$, or

   $b < a^2/4 + 2q$. 

\[1\]
Note that, in case $b > 0$, this is a stronger inequality than our previous inequality for the magnitude of $b$.

(10) Note that $\pi$ is a solution to the equation $X^2 - \beta X + q = 0$. Thus, in order for $\mathbb{Q}(\pi)$ to be a CM field of degree 4 over $\mathbb{Q}$, we need $\beta^2 - 4q$ under any embedding of $\mathbb{Q}(\beta)$ into $\mathbb{R}$ to be negative. In other words we need

$$\left(\frac{-a \pm \sqrt{\Delta}}{2}\right)^2 - 4q < 0$$

(11) We work out what this means:

\[
\begin{align*}
((−a ± \sqrt{\Delta})/2)^2 - 4q < 0 & \iff ((−a ± \sqrt{\Delta})/2)^2 < 4q \\
& \iff (−a ± \sqrt{\Delta})/2 < 2\sqrt{q} \text{ and } (−a ± \sqrt{\Delta})/2 > -2\sqrt{q} \\
& \iff (−a ± \sqrt{\Delta}) < 4\sqrt{q} \text{ and } (−a ± \sqrt{\Delta}) > -4\sqrt{q} \\
& \iff ±\sqrt{\Delta} < a + 4\sqrt{q} \text{ and } ±\sqrt{\Delta} > a - 4\sqrt{q} \\
& \iff \sqrt{\Delta} < a + 4\sqrt{q} \text{ and } -\sqrt{\Delta} > a - 4\sqrt{q} \\
& \iff \Delta < a^2 + 8a\sqrt{q} + 16q \text{ and } \Delta < a^2 - 8a\sqrt{q} + 16q \\
& \iff a^2 - 4b + 8q < a^2 - 8|a|\sqrt{q} + 16q \\
& \iff -4b < -8|a|\sqrt{q} + 8q \\
& \iff -b < -2|a|\sqrt{q} + 2q \\
& \iff b > 2|a|\sqrt{q} - 2q
\end{align*}
\]

Note that one of the conclusions of this sequence of inequalities is also that $|a| \leq 4\sqrt{q}$ which we saw before.

(12) So a complete set of inequalities is the following

\[
\begin{align*}
|a| & \leq 4\sqrt{q} \\
b & < a^2/4 + 2q \\
b & > 2|a|\sqrt{q} - 2q
\end{align*}
\]