Algebraic de Rham cohomology

Setting. Let k be a field of characteristic 0. For any scheme X of finite type over k we have the algebraic de Rham complex $\Omega^*_{X/k}$ of X over k. This is a complex of abelian groups whose terms are coherent sheaves on X. The algebraic de Rham cohomology of X is by definition the hyper cohomology of this complex:

$$H^*_{dR}(X) := \mathbf{H}^*(X, \Omega^*_{X/k}).$$

The hypercohomology of a bounded below complex of abelian sheaves is defined in the appendix.

Theorem. Assume k has characteristic 0. Algebraic de Rham cohomology is a Weil cohomology theory with coefficients in $K = k$ on smooth projective varieties over k.

We do not assume k algebraically closed since the most interesting case of this theorem is the case $k = \mathbf{Q}$. We will use the definition of Weil cohomology theories given in the note on Weil cohomology theories. Thus, in order for this to make sense I have to give you more of the data.

(D1) The first (D1) we have given above, except that we did not explain how to define the cupproduct. The cupproduct comes from wedge product on the de Rham complex:

$$\Omega^*_{X/k} \otimes_{\mathbf{Z}} \Omega^*_{X/k} \longrightarrow \Omega^*_{X/k},\ s \otimes t \mapsto s \wedge t.$$

The wedge product is graded commutative: if s is a local section of $\Omega^a_{X/k}$ and t is a local section of $\Omega^b_{X/k}$, then $s \wedge t = (-1)^{ab} t \wedge s$. Also, it is a derivation $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge d(t)$. It is these rules and the cup product in cohomology that gives rise to a graded commutative algebra structure on $H^*_{dR}(X)$. See appendix.

(D2) The pullback maps are defined by the functoriality of the de Rham complex: given a morphism $f : X \to Y$ we obtain a map of complexes

$$f^{-1}\Omega^*_Y \longrightarrow \Omega^*_X$$

which gives rise to a map $f^* : H^*_{dR}(X) \to H^*_{dR}(Y)$. This is a map of graded algebras because pulling back forms is compatible with wedge products.

Before we continue, we need to point out some properties of algebraic de Rham cohomology. In other words, we will first prove some of the axioms before introducing the trace map and cohomology classes. Note that the axioms of a Weil cohomology theory do not provide for the existence of cohomology groups defined for nonprojective varieties, but that we may use the fact that they are defined for de Rham cohomology in order to prove the axioms.

Hodge to de Rham spectral sequence. For any finite type X over k there is a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) => H^{p+q}_{dR}(X).$$

In particular, if X is affine then $H^i_{dR}(X) = H^i(\Gamma(X, \Omega^*_{X/k}))$. Going back to smooth projective X, the existence of this spectral sequence implies axioms (W1) "finite dimensionality" and (W2) "vanishing". Axiom (W3) "functoriality" is obvious.

De Rham cohomology of projective space. For projective space $\mathbf{P}^n$ we have $H^q(\mathbf{P}^n, \Omega^p) = 0$ if $p \neq q$ and 1-dimensional if $0 \leq p = q \leq n$. (See Harthorne Chapter III, Exercise 7.3.) Hence, still in the case of $\mathbf{P}^n$ the spectral sequence degenerates and we conclude that $H^{odd}_{dR}(\mathbf{P}^n) = 0$ and $H^{2i}_{dR}(\mathbf{P}^n)$ is 1-dimensional for $0 \leq i \leq n$.

De Rham cohomology of affine space. Another example that can be computed by hand is the de Rham cohomology of affine n-space $\mathbf{A}^n_k$ over k. Namely $H^i_{dR}(\mathbf{A}^n_k) = k$ for $i = 0$ and zero else.

Mayer-Vietoris spectral sequence. Another spectral sequence arises when $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of X. Namely, the Mayer-Vietoris spectral sequence

$$E_1^{p,q} = \oplus H^q_{dR}(U_{i_0 \ldots i_p}) => H^{p+q}_{dR}(X).$$

The construction of this spectral sequence is functorial in the following way. Suppose that $f : Y \to X$ is a morphism of schemes of finite type over k. Set $V_i = f^{-1}(U_i)$, so $\mathcal{V} : Y = \bigcup V_i$ is a covering of Y. Then $f^* : \oplus H^q_{dR}(U_{i_0 \ldots i_p}) \to \oplus H^q_{dR}(V_{i_0 \ldots i_p})$ extends to a morphism of spectral sequences. As an application, if $\mathcal{U}$ is an affine open covering of X, and if X is separated of finite type over k, then we deduce that the (simple) Cech complex $s\mathcal{C}^*(\mathcal{U}, \Omega^*_{X/k})$ computes the algebraic de Rham cohomology of X over k (see the appendix for the definition of this complex).

Künneth for affine varieties. Consider two projective nonsingular varieties X and Y over k. The product $X \times_{Spec(k)} Y$ is a smooth projective scheme over k. Its de Rham cohomology is the direct sum of the de Rham cohomology of its irreducible components. Hence, in order to prove that we have a Künneth decomposition we have to show that $H^*_{dR}(X \times Y)$ decomposes as the tensor product of $H^*_{dR}(X)$ and $H^*_{dR}(Y)$. This will take a bit of work. The first thing we use is that if A and B are finite type k-algebras, then

$$\Omega^1_{A \otimes_k B} = \Omega^1_{A/k} \otimes_k B \oplus A \otimes_k \Omega^1_{B/k}.$$

As a result we obtain a canonical identification of the de Rham complex of $A \otimes_k B$ over k as the complex associated to the double complex $\Omega^*_{A/k} \otimes \Omega^*_{B/k}$. Since there are no Tor groups to worry about as we're tensoring over a field, this proves the Künneth formula when X and Y are affine.

Cohomology of affine bundles. Next, suppose that n is an integer and $f : Y \to X$ is a morphism such that for every point $x \in X$ there exists a neighbourhood $x \in U \subset X$ such that $f^{-1}(U) \cong U \times \mathbf{A}^n$ as schemes over U. We will call such a morphism an affine bundle of dimension n. Note that vector bundles are affine bundles but in general an affine bundle need not have the structure of a vector bundle. In any case, if $f : Y \to X$ is an affine bundle then $f^* : H^*_{dR}(X) \to H^*_{dR}(Y)$ is an isomorphism. This can be seen by choosing an affine open covering $\mathcal{U} = \{U_i\}$ such that $f^{-1}(U_i) \cong U_i \times \mathbf{A}^n$ and considering the morphism of Mayer-Vietoris spectral sequences $f^* : \oplus H^q_{dR}(U_{i_0 \ldots i_p}) \to \oplus H^q_{dR}(V_{i_0 \ldots i_p})$. By the above, we see that each map $H^q_{dR}(U_{i_0 \ldots i_p}) \to H^q_{dR}(V_{i_0 \ldots i_p})$ is an isomorphism because of Künneth in the affine case and the computation of $H^*_{dR}(\mathbf{A}^n)$. We will use the result by Jouanolou that if X is quasi-projective, then there exists an integer n and an affine bundle $X' \to X$ of dimension n such that X' is *affine*. See Proposition 1.1.3 in "Lectures on Algebro-Geometric Chern-Weil and Cheeger-Chern-Simons Theory for Vector Bundles" by Bloch and Esnault.

Künneth for quasi-projective schemes over k. We can now prove the Küneth when X and Y are quasi-projective as follows. Choose affine bundles $X' \to X$ and $Y' \to Y$ such that X' and Y' are affine. Consider the commutative diagram

$$\begin{array}{ccc} H^*_{dR}(X) \otimes H^*_{dR}(Y) & \longrightarrow & H^*_{dR}(X \times Y) \\ \downarrow & & \downarrow \\ H^*_{dR}(X') \otimes H^*_{dR}(Y') & \longrightarrow & H^*_{dR}(X' \times Y') \end{array}$$

Note that $X' \times Y' \to X \times Y$ is an affine bundle as well. Thus the Künneth decomposition for $X \times Y$ follows from the Künneth decomposition for the affine case. So finally (W4) follows.

Remark. Of course the above argument employs a trick. The Künneth decomposition holds in general. One way to prove it is to use the functoriality of the Mayer-Vietoris spectral sequence with regards to the projection maps $X \times Y \to Y$ and an affine open covering of Y. In order to make this work I think you have to show the resulting Mayer-Vietoris spectral sequence for $X \times Y$ is a spectral sequence in the category of $H^*_{dR}(X)$-modules. This takes a bit more work...

Trace map for finite morphisms of smooth varieties. There is a nice construction of a trace map for a finite morphism of smooth projective varieties $f : X \to Y$ of the same dimension. Namely, there is an obvious map

$$i^p : f_*\mathcal{O}_X \otimes \Omega^p_{Y/k} \to f_*\Omega^p_{X/k}.$$

It turns out that there is a unique map

$$\Theta^p : f_*\Omega^p_{X/k} \to \Omega^p_{Y/k}$$

such that $\Theta^p \circ i^p = Tr \otimes id$, where Tr indicates the trace map $Tr : f_*\mathcal{O}_X \to \mathcal{O}_Y$. You can either show that Θ^p exists directly (by looking in codimension 1 on Y) or you can look at the article "An extension of the trace map" by Emmanuelle Garel, where it is defined in a more general setting. Note that some assumptions on the singularities of X and Y is necessary. It turns out (and it is easy to show this in the case of smooth X and Y) that Θ^* is a map of complexes

$$\Theta^* : f_*\Omega^*_{X/k} \longrightarrow \Omega^*_{Y/k}$$

and that the composition

$$\Omega^*_{Y/k} \longrightarrow f_*\Omega^*_{X/k} \longrightarrow \Omega^*_{Y/k}$$

is multiplication by the degree of the finite map. In particular this implies that if $f : X \to Y$ is a finite morphism of smooth projective varieties of the same dimension then f^* is injective.

Some properties of the trace map. Continuing with notation as above, note that the complexes $f_*\Omega^*_{X/k}$ and $\Omega^*_{X/k}$ are sheaves differential graded commutative algebras. The map $\Omega^*_{Y/k} \to f_*\Omega^*_{X/k}$ is a map sheaves of graded commutative differential algebras and it gives rise to the map f^* on cohomology. Note that Θ^* is not a map of sheaves of graded commutative differential algebras. It is a map of graded modules over $\Omega^*_{Y/k}$. Therefore, indicating $\Theta : H^*_{dR}(X) \to H^*_{dR}(Y)$ the induced map on cohomology we deduce that $\Theta(\alpha) \cup \beta = \Theta(\alpha \cup f^*\beta)$.

Ring structure on the de Rham cohomology of projective space. Using this we can compute the ring structure on $H^*_{dR}(\mathbf{P}^n)$. Pick any nonzero element $h \in H^2_{dR}(\mathbf{P}^1)$. By degree reasons we have $H^*_{dR}(\mathbf{P}^1) = k[h]/(h^2)$. By Künneth it follows that $H^*_{dR}((\mathbf{P}^1)^n) = k[h_1, \ldots, h_n]/(h_i^2)$. The symmetric group S_n acts via permutation of the h_i. (There are no sign issues since all classes are in even degrees.) Consider the S_n-equivariant finite map $(\mathbf{P}^1)^n \to \mathbf{P}^n$ where S_n acts trivially on $\mathbf{P}^n$, see Exercise 17 of the note on Weil cohomology. Note that the compostion$\mathbf{P}^1 \to \mathbf{P}^1 \times 0 \times \ldots \times 0 \to (\mathbf{P}^1)^n \to \mathbf{P}^n$ is the standard embedding $\mathbf{P}^1 \to \mathbf{P}^n, x \mapsto (x, 0, \ldots, 0)$. The injectivity of $H^*_{dR}(\mathbf{P}^n) \to H^*_{dR}((\mathbf{P}^1)^n)$, combined with the S_n-equivariance implies that $H^*_{dR}(\mathbf{P}^n) \cong k[h]/(h^{n+1})$ with (symbolically) $h = h_1 + \ldots + h_n$.

A mildly generalized Poincaré duality. Let X be a projective d-dimensional Gorenstein variety. This means the dualizing sheaf $\omega = \omega_{X/k}$ is an invertible $\mathcal{O}_X$-module, there is a trace map $H^d(X, \omega) \to k$, and there is a perfect duality paring $H^i(X, \mathcal{F}) \times Ext^{d-i}_{\mathcal{O}_X}(\mathcal{F}, \omega) \to H^d(X, \omega) \to k$ for every coherent sheaf $\mathcal{F}$. Let $\mathcal{F}^*$, $\mathcal{G}^*$, and $\mathcal{H}^*$ be complexes of sheaves of k-vector spaces on X and let

$$\gamma : Tot(\mathcal{F}^* \otimes_k \mathcal{G}^*) \longrightarrow \mathcal{H}^*$$

be a map of comlexes of sheaves with the following properties:

(1) We have $\mathcal{H}^i = 0$ for $i > 0$,
(2) $\mathcal{H}^0 \cong \omega$,
(3) the map $\omega[0] \to \mathcal{H}^*$ induces an isomorphism $H^d(X, \omega) = \mathbf{H}^d(X, \mathcal{H}^*)$,
(4) there exist $a \leq b$ such that $\mathcal{F}^i$ is zero unless $i \in [a, b]$ and $\mathcal{G}^j = 0$ unless $j \in [-b, -a]$,
(5) each of the sheaves $\mathcal{F}^i$ and $\mathcal{G}^j$ can be given the structure of a locally free $\mathcal{O}_X$-module (but the differentials are not necessarily $\mathcal{O}_X$ linear) such that for each $i \in [a, b]$ the pairing γ induces a perfect $\mathcal{O}_X$-linear pairing of $\mathcal{O}_X$-modules $\mathcal{F}^i \otimes \mathcal{G}^{-i} \to \omega$. In other words, it induces an isomorphism $\mathcal{F}^i \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}^{-i}, \omega)$.

We claim this kind of structure gives rise to a perfect pairing

$$\mathbf{H}^i(X, \mathcal{F}^*) \times \mathbf{H}^{d-i}(X, \mathcal{G}^*) \longrightarrow \mathbf{H}^d(X, \mathcal{H}^*) \cong H^d(X, \omega) \to k$$

To prove this, we may argue by induction on $b - a$. The case $b - a = 0$ holds by assumption that X be Gorenstein. For the general case we consider the following exact sequences of complexes

$$0 \to \mathcal{F}^b[-b] \to \mathcal{F}^* \to \mathcal{F}^{\leq b-1} \to 0, \text{ and } 0 \to \mathcal{G}^{\geq -b+1} \to \mathcal{G}^* \to \mathcal{G}^{-b}[b] \to 0$$

which give exact sequences

$$\mathbf{H}^{i-1}(X, \mathcal{F}^{\leq b-1}) \xrightarrow{\alpha} \mathbf{H}^i(X, \mathcal{F}^b[-b]) \xrightarrow{\beta} \mathbf{H}^i(X, \mathcal{F}^*) \xrightarrow{\gamma} \mathbf{H}^i(X, \mathcal{F}^{\leq b-1}) \xrightarrow{\delta} \mathbf{H}^{i+1}(X, \mathcal{F}^b[-b])$$

and

$$\mathbf{H}^{j+1}(X, \mathcal{G}^{\geq -b+1}) \xleftarrow{\alpha^t} \mathbf{H}^j(X, \mathcal{G}^{-b}[b]) \xleftarrow{\beta^t} \mathbf{H}^j(X, \mathcal{G}^*) \xleftarrow{\gamma^t} \mathbf{H}^j(X, \mathcal{G}^{\geq -b+1}) \xleftarrow{\delta^t} \mathbf{H}^{j-1}(X, \mathcal{G}^{-b}[b])$$

Of course, we are especially interested in the case $j = d - i$ which is what assume now. Note that the complexes $\mathcal{F}^{\leq b-1}$, $\mathcal{G}^{\geq -b+1}$, $\mathcal{H}^*$ combined with the restriction of γ form another datum satisfying (1)–(5) above. Hence by induction the pairing between $\mathbf{H}^\ell(X, \mathcal{F}^{\leq b-1})$ and $\mathbf{H}^{d-\ell}(X, \mathcal{G}^{\geq -b+1})$ is perfect for any ℓ. By duality on X the γ-induced pairing between $\mathbf{H}^\ell(X, \mathcal{F}^b[-b])$ and $\mathbf{H}^{d-\ell}(X, \mathcal{G}^{-b}[b])$ is perfect for any ℓ as well. Now it is fairly easy to see that with these pairings we have that $\langle \beta x, y\rangle = \langle x, \beta^t y\rangle$ for all $x \in \mathbf{H}^i(X, \mathcal{F}^b[-b])$ and $y \in \mathbf{H}^j(X, \mathcal{G}^*)$. Similarly for γ and γ^t. In order to conclude from this we really also need to show that (α, α^t) and (δ, δ^t) are adjoint pairs up to sign. This is shown in the appendix on the cup product, and hence the claim follows.

Poincaré duality. Let X be a smooth projective variety over k of dimension d. We verify the hypotheses (1)–(5) above for the triple

$$(\Omega^*_{X/k}, \Omega^*_{X/k}[d], \Omega^*_{X/k}[d], \wedge).$$

Property (1) is clear. Since X is smooth and projective we have $\omega_{X/k} \cong \Omega^d_{X/k}$, and X is Gorenstein. For a proof of this fact see for example Hartshorne, Chapter III, Corollary 7.12. This proves (2). To prove (3) we have to show that the natural map $H^d(X, \omega_{X/k}) = H^d(X, \Omega^d_{X/k} \to H^{2d}_{dR}(X, \Omega^*_{X/k})$ is an isomoprhism. Consider the Hodge to de Rham spectral sequence for X. The only differential in the Hodge to de Rham spectral sequence that can affect $H^{2d}_{dR}(X)$ is $d_1^{d,d-1} : H^d(X, \Omega^{d-1}_{X/k}) \to H^d(X, \Omega^d_{X/k})$. By Serre duality, $H^d(X, \Omega^n_{X/k})$ is a free rank 1 $H^0(X, \mathcal{O}_X)$-module. The image of $d_1^{d,d-1}$ is a $H^0(X, \mathcal{O}_X)$-submodule. Thus it suffices to show that $H^d_{dR}(X)$ is not zero. Choose a finite surjective morphism $\pi : X \to \mathbf{P}^d$. By the trace map for π introduced above we conclude that $H^{2d}_{dR}(X) \neq 0$ because it contains a copy of $H^{2d}_{dR}(\mathbf{P}^d)$. This proves (3). Statements (4) and (5) are obvious. In particular axiom (W5) for "untwisted" algebraic de Rham cohomology follows for any choice of isomorphism $\omega_{X/k} \cong \Omega^n_{X/k}$, which induces a nonzero k-linear map $H^{2d}_{dR}(X) \to k$.

Remark. But in fact any nonzero k-linear map $\gamma : H^{2d}_{dR}(X) \to k$ will give rise to a Poincaré duality for $H^*_{dR}(X)$. This is so because $H^{2d}_{dR}(X)$ is a free rank 1 $H^0(X, \mathcal{O}_X)$-module, and any two such linear maps γ and γ' differ by multiplication by a unit u in the field $H^0(X, \mathcal{O}_X)$. Hence the two pairings $\langle , \rangle$ and $\langle , \rangle'$ on algebraic de Rham cohomology will be related by the formula $\langle \alpha, \beta\rangle' = \langle \alpha, u\beta\rangle$. Whence the result.

(D3) Tate twists. We set $K(-1) = H^2_{dR}(\mathbf{P}^1_k)$, and we deduce all $K(n)$ from this by tensor constructions.

Normalization of the cohomology groups of all projective spaces. Pick (temporarily) a generator h of the 1-dimensional vector space $K(-1)$. We claim that for every $n \geq 1$ there is a unique $h \in H^2_{dR}(\mathbf{P}^n)$ such that
(i) for every linear map $\mathbf{P}^n \to \mathbf{P}^m$ the class h pulls back to h, and
(ii) the Segre map $\mathbf{P}^n \times \mathbf{P}^m \longrightarrow \mathbf{P}^{nm+n+m}$ pulls h back to $1 \otimes h + h \otimes 1$.
To see this, because we have Künneth, it is enough to show that for any pair of linear maps $a, b : \mathbf{P}^1 \to \mathbf{P}^n$ the pullback maps a^* and b^* are the same. (Note: we have seen prevously that the pull back map for the standard line in $\mathbf{P}^n$ is nonzero.) But any linear map is corresponds to a point in the smooth connected variety parametrizing linear embeddings of $\mathbf{P}^1$ into $\mathbf{P}^n$. The claim follows from the fact below.

Homotopy invariance. Suppose that X, T, Y are quasi-projective varieties over k and that $f : X \times T \to Y$ is a morphism of schemes over k. Suppose that T is smooth and connected. Then for any pair of k-points $a, b \in T(k)$ the induced maps $X \to X \times a \to Y$ and $X \to X \times b \to Y$ induce the same map on cohomology. This follows immediately from Künneth for $X \times T$ and the fact that $H^0_{dR}(T) = k$ under the assumptions given (note that the existence of $a \in T(k)$ implies that T is geometrically irreducible, and hence the only "constant" global functions on T correspond to the elements of k).

The twisted cohomology of $\mathbf{P}^n$. We collect the information obtained above as follows

$$\bigoplus\nolimits_{i=0}^{n} H^{2i}_{dR}(\mathbf{P}^n)(i) = k[\xi]/(\xi^{n+1})$$

where $\xi \in H^2_{dR}(\mathbf{P}^n)(1) = Hom_k(H^2_{dR}(\mathbf{P}^1), H^2_{dR}(\mathbf{P}^n))$ is the *inverse* of the pullback map. In terms of our choice of h upstairs this is the element $h \otimes h^*$; use this to verify some of the claims below. By the above, under the Segre map ξ maps to $\xi \otimes 1 + 1 \otimes \xi$.

Chern classes of invertible sheaves. We use the following method to define Chern classes of invertible sheaves

$$c_1^{dR} : Pic(X) \to H^2(X)(1).$$

This will be a homomorphism of abelian groups functorial in X. First of all, we define

$$c_1^{dR}(\mathcal{O}_{\mathbf{P}^n}(1)) = \xi.$$

For any quasi-projective X, any invertible sheaf $\mathcal{L}$ and any morphism $f : X \to \mathbf{P}^n$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ we define $c_1^{dR}(\mathcal{L}) = c_1^{dR}(f^*\mathcal{O}(1)) = f^*\xi$. This is independent of the choice of f by the results on linear maps of projective spaces above. In particular, this defines a chern class for any very ample invertible sheaf. Since on a quasi-projective X every invertible sheaf can be written as the difference of ample invertible sheaf this defines c_1^{dR} in general. To see that c_1^{dR} is well defined, we have to verify that $c_1^{dR}(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1^{dR}(\mathcal{L}_1) + c_1^{dR}(\mathcal{L}_2)$ for semi-ample invertible sheaves $\mathcal{L}_i$. This follows from our results on the Segre maps. By construction, this chern class is functorial with respect to pullbacks.

Cohomology of projective space bundles. Consider a quasi-projective variety X and a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Our convention is that

$$\mathbf{P}(\mathcal{E}) = Proj(Sym^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E}}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Let $c = c_1^{dR}(\mathcal{O}_{\mathbf{P}(\mathcal{E}}(1)) \in H^2_{dR}(\mathbf{P}(\mathcal{E}))(1)$. We claim that

$$\bigoplus\nolimits_{i=0}^{r-1} H^*_{dR}(X)(-i) \longrightarrow H^*_{dR}(\mathbf{P}(\mathcal{E})),\ (\alpha_0, \ldots, \alpha_{r-1}) \mapsto \alpha_0 + \alpha_1 \cup c + \ldots + \alpha_{r-1} \cup c^{r-1}$$

is an isomorphism where $r = rank(\mathcal{E})$. When X is affine and $\mathcal{E}$ is trivial this holds by Künneth. In general, choose an affine open covering of X such that $\mathcal{E}$ is trivial on the members and use the functoriality of the Mayer-Vietoris spectral sequence. (Actually, it is probably easier to use induction on the number of such affine opens needed to cover X and to use the "usual" Mayer-Vietoris long exact sequence for a covering by two opens.) Details left as an exercise.

Chern classes of vector bundles. Let X be a quasi-projective scheme over k. We define the de Rham Chern classes of a finite locally free sheaf $\mathcal{E}$ of rank r as follows. They are the elements $c_i^{dR}(\mathcal{E}) \in H^{2i}_{dR}(X)(i)$, $i = 0, \ldots r$ such that $c_0^{dR}(\mathcal{E}) = 1$, and

$$-c^r = \sum\nolimits_{i=1}^{r} (-1)^i c_i^{dR}(\mathcal{E}) \cup c^{r-i}.$$

where $c = c_1^{dR}(\mathcal{O}(1)_{\mathbf{P}(\mathcal{E})})$. This makes sense by our computation of the cohomology of a projective space bundle above. These chern classes are obviously compatible with pullbacks. Note that by our conventions, the first chern class of an invertible sheaf is unchanged. The *total chern class* of $\mathcal{E}$ is the element

$$c^{dR}(\mathcal{E}) = c_0^{dR}(\mathcal{E}) + c_1^{dR}(\mathcal{E}) + \ldots + c_r^{dR}(\mathcal{E})$$

At this point there is one trivial remark we can make. Suppose that $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}$. In this case there is a canonical morphsm

$$\begin{array}{ccc} \mathbf{P}(\mathcal{E}) & \xrightarrow{\ \ g\ \ } & \mathbf{P}(\mathcal{E}') \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

such that $g^*\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})} \otimes \pi^*\mathcal{L}$. It follows easily from this that $c_i^{dR}(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^{i} c_j^{dR}(\mathcal{E}) \cup c_1^{dR}(\mathcal{L})^j$.

The chern classes of the sum of invertible sheaves. Let X be a quasi-projective scheme and let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Set $c_1^{dR}(\mathcal{L}_i) = x_i$. We claim that

$$c^{dR}(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r) = \prod\nolimits_{i=1}^{r}(1 + x_i)$$

By the last property of the section definiting the chern classes we see that we may tensor all $\mathcal{L}_i$ with some hugely ample invertible $\mathcal{O}_X$-module. Hence we may assume that there are morphisms $f_i : X \to \mathbf{P}^{n_i}$ (with $n_i >> 0$) such that $\mathcal{L}_i = f_i^*(\mathcal{O}(1)$. In other words, we have reduced our claim to the case that $X = \prod_{i=1}^{r} \mathbf{P}^{n_i}$, and $\mathcal{L}_i = pr_i^*\mathcal{O}(1)$. The projection $\mathcal{E} := \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r \to \mathcal{L}_i$ gives rise to a unique section

$$\sigma_i : X \to \mathbf{P}(\mathcal{E}))$$

such that $\sigma_i^*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = \mathcal{L}_i$. Thus the chern classes $c_j = c_j^{dR}(\mathcal{E})$ satisfy the relations

$$-x_i^r = \sum\nolimits_{j=1}^{r}(-1)^j c_j x_i^{r-j}$$

for all $i = 1, \ldots r$. Note that this relation says that x_i is a root of the polynomial $T^r - c_1T^{r-1} + \ldots + (-1)^r c_r$. Since the cohomology ring of X is basically just the polynomial ring $k[x_1, \ldots, x_r]$ (some relations occur but they are sitting in high degrees), we obtain the desired result by some simple algebra.

Splitting principle. For any finite locally free sheaf $\mathcal{E}$ on any quasi-projective scheme X over k there exists a smooth quasi-projective morphism $f : Y \to X$ such that

$f^* : H^*_{dR}(X) \to H^*_{dR}(Y)$ is injective, and

$f^*\mathcal{E} \cong \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r$ for some invertible $\mathcal{O}_Y$-modules $\mathcal{L}_i$.

To prove this we first reduce to the case where $\mathcal{E}$ has a finite filtration F^* whose graded pieces are invertible $\mathcal{O}_X$-modules. For example one pull back $\mathcal{E}$ the total flag bundle of $\mathcal{E}$. This choice has the benefit that we've already shown the induced map on de Rham cohomology $H^*_{dR}(X) \to H^*_{dR}(Y)$ is injective since it is a repeated projective space bundle. After this, note that given a surjection of finite locally free $\mathcal{O}_X$-modules $\mathcal{E} \to \mathcal{F}$ the space of sections is an affine bundle over X. Hence the result follows.

Additivity of chern classes. Suppose that $\mathcal{E}$ sits in an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

We claim that $c^{dR}(\mathcal{E}) = c^{dR}(\mathcal{E}_1) \cup c^{dR}(\mathcal{E}_2)$. The space of splittings of the exact sequence is an affine bundle over X and hence we may assume that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Next, by the splitting principle, we may assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are direct sums of invertible sheaves. In this case the result follows from our calculation of the chern class of a direct sum of invertible sheaves.

Splitting principle revisited. Using the above we formally write

$$c^{dR}(\mathcal{E}) = \prod(1 + x_i)$$

and we call x_i the *Chern roots* of $\mathcal{E}$. Of course it really doesn't make sense in the cohomology of X, but it does make sense in some other variety Y such that the cohomology of X injects into it. As is customary, any symmetric polynomial in the x_i then corresponds to a cohomology class on X because the chern classes of $\mathcal{E}$ are up to sign the elementary symmetric functions in the x_i.

Remark. It is especially nice to work in the cohomology of the Flag variety of $\mathcal{E}$ because $\bigoplus H^{2i}(Flag\mathcal{E})(i) = \big(\bigoplus H^{2i}_{dR}(X)(i)\big)[x_1, \ldots, x_r]/I$ where I is smallest ideal such that the following equation holds true: $\prod_{i=1}^{r}(T - x_i) = T^r - c_1^{dR}(\mathcal{E})T^{r-1} + \ldots + (-1)^r c_r(\mathcal{E})$. This can be proved by repeated application of the projective space bundle formula. (And of course there is a corresponding statement for the odd cohomology.)

Chern classes and tensor product. We define the *Chern character* of a finite locally free sheaf of rank r to be the expression

$$ch^{dR}(\mathcal{E}) := \sum\nolimits_{i=1}^{r} e^{x_i}$$

if the x_i are the chern roots of $\mathcal{E}$. By the above we have, in case of an exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ that $ch^{dR}(\mathcal{E}) = ch^{dR}(\mathcal{E}_1) + ch^{dR}(\mathcal{E}_2)$. Using the Chern character we can express the compatibility of the chern classes and tensor product as follows:

$$ch^{dR}(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch^{dR}(\mathcal{E}_1) \cup ch^{dR}(\mathcal{E}_2)$$

The proof follows directly from the splitting principle.

Tiny bit of K-theory. Let X be a *smooth* quasi-projective variety over k. We will use the following facts:
(1) For any coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists a finite complex of finite locally free $\mathcal{O}_X$-modules $\mathcal{F}^*$ and a quasi-isomorphism $\mathcal{F}^* \to \mathcal{F}[0]$. We will call such a quasi-isomorphism $\mathcal{F}^* \to \mathcal{F}[0]$ a *resolution* of $\mathcal{F}$.
(2) For any short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ there exist resolutions $\mathcal{F}_i^* \to \mathcal{F}_i[0]$ and a short exact sequence $0 \to \mathcal{F}_1^* \to \mathcal{F}_2^* \to \mathcal{F}_3^* \to 0$ of complexes that recovers the short exact sequence upon taking cohomology sheaves.

In these statements it is convenient to have complexes supported in degrees ≤ 0 but it is not necessary. Basically, (1) and (2) follow from the following two statements: (a) for every coherent sheaf $\mathcal{F}$ on X there exists a direct sum $\oplus\mathcal{O}_X(-n)$ which surjects onto $\mathcal{F}$, and (b) given an exact complex $0 \to \mathcal{G} \to \mathcal{F}_N \to \ldots \to \mathcal{F}_0 \to \mathcal{F} \to 0$ with $\mathcal{F}$ coherent, and $\mathcal{F}_i$ finite locally free, and $N \geq \dim X - 1$ then $\mathcal{G}$ is locally free. The first is standard, see Hartshorne, Chapter II, Corollary 5.18. The second is Serre's $pd + depth = dim$, see Matsumura, Commutative Algebra, page 113.

For any variety X we define two abelian groups $K^0(X)$ and $K_0(X)$. The group $K^0(X)$ is the free abelian group generated by finite locally free $\mathcal{O}_X$-modules modulo the relation that $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$ whenever $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules. The group $K_0(X)$ is the free abelian group generated by coherent $\mathcal{O}_X$-modules modulo the relation that $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ whenever $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of coherent $\mathcal{O}_X$-modules. These relations signify that if $\mathcal{E}^*$ (resp. $\mathcal{F}^*$) is a finite *exact* complex of finite locally free (resp. coherent) $\mathcal{O}_X$-modules then $\sum(-1)^i[\mathcal{E}_i] = 0$ (resp. $\sum(-1)^i[\mathcal{F}_i] = 0$) in $K^0(X)$ (resp. $K_0(X)$). These groups satisfy the following functorialities

If $f : X \to Y$ is any morphism then there is a map $f^* : K^0(Y) \to K^0(X)$ defined by pullback.

If $f : X \to Y$ is a flat morphism of schemes of finite type over k there is a map $f^* : K_0(Y) \to K_0(X)$ defined by pullback.

If $f : X \to Y$ is a proper morphism of finite type schemes over k there is a map $f_* : K_0(X) \to K_0(Y)$ defined by the rule $f_*([\mathcal{F}]) = \sum_i(-1)^i[R^if_*(\mathcal{F})]$.

Finally, there is an obvious homomorphism of abelian groups

$$K^0(X) \longrightarrow K_0(X).$$

Claim: If X is a *smooth* quasi-projective variety then this homomorphism is an isomorphism. As an inverse we map the class $[\mathcal{F}]$ in $K_0(X)$ to $I(\mathcal{F}) := \sum(-1)^i[\mathcal{F}^i]$ in $K^0(X)$, if $\mathcal{F}^* \to \mathcal{F}[0]$ is a resolution as above. The main problem is to show that this is well defined. Let us say that a coherent $\mathcal{O}_X$-module has property $\mathcal{P}$ if this is the case. A finite locally free $\mathcal{O}_X$-module has property $\mathcal{P}$ as is clear from the definition of $K^0(X)$. Furthermore, assertion (2) above says that if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of coherent $\mathcal{O}_X$-modules having property $\mathcal{P}$, then $I(\mathcal{F}_2) = I(\mathcal{F}_1) + I(\mathcal{F}_3)$. We argue that every coherent $\mathcal{O}_X$-module has property $\mathcal{P}$ by induction on the projective dimension of the sheaf $\mathcal{F}$. Suppose that $\mathcal{F}_1^* \to \mathcal{F}[0]$ and

$\mathcal{F}_2^* \to \mathcal{F}[0]$ are two resolutions of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\mathcal{F}_i^j = 0$ for $j > 0$. Consider the sheaf $\mathcal{F}_{12}$ fitting into the following exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{K}_2 & = & \mathcal{K}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}_1 & \to & \mathcal{F}_{12} & \to & \mathcal{F}_2^0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}_1 & \to & \mathcal{F}_1^0 & \to & \mathcal{F} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathcal{K}_i$ is the kernel of $\mathcal{F}_i^0 \to \mathcal{F}$. Note that the projective dimensions of $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{F}_{12}$ are less than the projective dimension of $\mathcal{F}$ (unless $\mathcal{F}$ was locally free to begin with). Hence, our induction hypothesis applies to these sheaves. In particular we see that $I(\mathcal{K}_1) + I(\mathcal{F}_2^0) = I(\mathcal{K}_2) + I(\mathcal{F}_1^0)$. Since we may use the complex $\mathcal{F}_i^{\leq -1}[1]$ as a resolution of $\mathcal{K}_i$ we have $I(\mathcal{K}_i) = \sum (-1)^{j+1}[\mathcal{F}_i^j]$. The desired result $\sum_i (-1)^i [\mathcal{F}_1^i] = \sum_i (-1)^i [\mathcal{F}_2^i]$ follows.

The Chern character of a coherent sheaf. For any quasi-projective variety X over k we can extend the Chern character to a homomorphism

$$ch^{dR} : K^0(X) \longrightarrow \bigoplus H^{2i}_{dR}(X)(i).$$

In the *smooth* case we use the isomorphism $K^0(X) \to K_0(X)$ to define the *Chern character* of a coherent $\mathcal{O}_X$-module $\mathcal{F}$. The recipe is that we choose a resolution $\mathcal{F}^* \to \mathcal{F}$ as in (1) and we set

$$ch^{dR}(\mathcal{F}) := ch^{dR}(\mathcal{F}^*) := \sum\nolimits_i (-1) ch^{dR}(\mathcal{F}^i).$$

This is well defined because of the isomorphism $K^0(X) \to K_0(X)$ above. We have additivity for this chern character. Namely, suppose that $0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F} \to 0$ is a short exact sequence of coherent sheaves on X. Then $ch^{dR}(\mathcal{G}) = ch^{dR}(\mathcal{H}) + ch^{dR}(\mathcal{F})$. In general we no longer have $ch^{dR}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = ch^{dR}(\mathcal{F}) \cup ch^{dR}(\mathcal{G})$ for a pair of coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on X. The correct statement is

$$ch^{dR}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) = \sum\nolimits_{i=0}^{\dim X} (-1)^i ch^{dR}(Tor_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = ch^{dR}(\mathcal{F}) \cup ch^{dR}(\mathcal{G}).$$

The *derived* tensor product $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ can be computed by taking a resolution $\mathcal{F}^* \to \mathcal{F}[0]$ and taking the complex $\mathcal{F}^* \otimes_{\mathcal{O}_X} \mathcal{G}$. A final obvious fact is that if $f : X \to Y$ is a *flat* morphism of smooth quasi-projective varieties over k then $f^*(ch^{dR}(\mathcal{F})) = ch^{dR}(f^*\mathcal{F})$ for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$. In fact, it suffices that f is flat at all points of $f^{-1}(Supp(\mathcal{F}))$. The reason is that in this case a resolution will pull back to a resolution.

Example. The cohomology class of a skyscraper sheaf on $\mathbf{P}^n$. Let $p \in \mathbf{P}^n(k)$ be a rational point and denote k_p the skyscraper sheaf at the point p. There is a finite locally free resolution

$$0 \to \mathcal{O}(-n) \to \mathcal{O}(-n+1)^n \to \ldots \to \mathcal{O}(-i)^{\binom{n}{i}} \to \ldots \to \mathcal{O}(-1)^n \to \mathcal{O} \to k_p \to 0.$$

By definition we get

$$ch^{dR}(k_p) = \sum\nolimits_{i=0}^n (-1)^i ch^{dR}(\mathcal{O}(-i)^{\binom{n}{i}}) = \sum\nolimits_{i=0}^n (-1)^i \binom{n}{i} e^{-i\xi} = (1 - e^{-\xi})^n = \xi^n$$

because $\xi^{n+1} = 0$.

Chern character in flat families. Suppose that T is a smooth quasi-projective connected variety over k. Let X be a smooth projective variety over k. Suppose $\mathcal{F}$ is a coherent $\mathcal{O}_{X\times T}$-module which is flat over T. Finally, let $a, b \in T(k)$ be rational points and let $\mathcal{F}_a$ be the pullback of $\mathcal{F}$ via the morphism $X \to X \times a \to X \times T$, and similarly for $\mathcal{F}_b$. We claim that

$$ch^{dR}(\mathcal{F}_a) = ch^{dR}(\mathcal{F}_b).$$

This would be completely clear if $\mathcal{F}$ were a locally free $\mathcal{O}_{X\times T}$-module, by an argument similar to the one proving "homotopy invariance". In general, flatness of $\mathcal{F}$ over T implies that if $\mathcal{F}^* \to \mathcal{F}[0]$ is a resolution (as in (1) above) then $\mathcal{F}_a^* \to \mathcal{F}_a$ is a resolution as well. Detail left to the reader.

The Chern character of a skyscraper sheaf. Suppose that x, x' are closed points in the smooth quasi-projective variety X over k. Set $d = \deg_k(x)$ and $d' = \deg_k(x')$. Denote $\kappa(x)$, $\kappa(x')$ the skyscraper sheaves (of residue fields). We claim that

$$d' \; ch^{dR}(\kappa(x)) = d \; ch^{dR}(\kappa(x')).$$

Of course this follows from the above because the family of all zero cycles of a given degree form an irreducible family, and we have invariance of Chern character in flat families. The details are slightly harder than you would think at first since we do not assume a variety is geometrically irreducible. Observe that the structure morphism $X \to Spec(k)$ factors as

$$X \longrightarrow Spec(k') \longrightarrow Spec(k)$$

where k' is the algebraic closure of k in the function field $k(X)$. Recall that $k' \supset k$ is a finite extension. Set $e = [k' : k]$. Since the characteristic of k is zero, since X is irreducible, and since X is smooth over k we conclude that X is geometrically irrreducible over k'. Hence we can find a smooth curve $C \subset X$ with $x, x' \in C$ which is geometrically irreducible over k' as well. (Using Bertini over k, see the excellent book "Theoremes de Bertini et Applications" by Jouanolou.) In this case x and x' give rise to effective Cartier divisors (x') and (x) on C of degree d/e, respectively d'/e over k'. Then $d'(x)$ and $d(x')$ correspond to k'-points a', b' of the smooth geometrically irreducible variety $Sym_{k'}^{dd'/e}(C)$ over k'. Consider the restriction of scalars

$$T := Res_{k'/k}\Big(Sym_{k'}^{dd'/e}(C)\Big).$$

A R-point t of T is by definition a $R \otimes k'$ point of $Sym_{k'}^{dd'/e}(C)$ for any k-algebra R. In particular there is a canonical morphism $T \times_{Spec(k)} Spec(k') \to Sym_{k'}^{dd'/e}(C)$. From general properties of restrictions of scalars we get that T is smooth and geometrically irreducible over k. Also, a', b' correspond to k-points a, b of T. Let $D' \subset Sym_{k'}^{dd'/e}(C) \times_{Spec(k')} C$ be the universal degree dd'/e divisor over k'. Let $D \subset T \times_{Spec(k)} C$ be the divisor which is the inverse image of D' under

$$T \times_{Spec(k)} C = T \times_{Spec(k)} Spec(k') \times_{Spec(k')} C \longrightarrow Sym_{k'}^{dd'/e}(C) \times_{Spec(k')} C$$

Then D is a flat family of closed subschemes of C of length dd' over k and $D_a = d'(x)$ and $D_b = d(x')$ as closed subschemes over k. Thus we conclude that $ch^{dR}(\mathcal{O}_{d'(x)}) = ch^{dR}(\mathcal{O}_{d(x')})$ in the cohomology of X. We leave it as an exercise to show that $ch^{dR}(\mathcal{O}_{d'(x)}) = d' ch^{dR}(\kappa(x))$ and similarly for the other side. (Hint: Use additivity of the Chern character.)

The cohomology class of a point. The upshot is that the element

$$u_X := \frac{1}{\deg_k(x)} ch^{dR}(\kappa(x))$$

for any closed point $x \in X$ is a canonical element of $H^{2\dim X}(X)(\dim X)$. If $f : X \to Y$ is a generically finite morphism of smooth projective varieties of the *same* dimension then $f^* u_Y = \deg(f) u_X$. Namely we can find a point $y \in Y$ such that $f^{-1}(y) = \{x_1, \ldots, x_t\}$ is finite and such that f is flat and unramified in all points of $f^{-1}(y)$ (uses characteristic zero). Since in this case $\deg(f)\deg_k(y) = \sum \deg_k(x_i)$ we conclude.

Nonvanishing. Sofar we have not argued that u_X is nonzero. The reason that it is nonvanishing is that it is equal to $\xi^n \neq 0$ in the case of $\mathbf{P}^n$, and by the fact that choosing a finite surjective morphism $\pi : X \to \mathbf{P}^{\dim X}$ gives an *injection* $H^*_{dR}(\mathbf{P}^{\dim X}) \to H^*_{dR}(X)$ (by the trace map for π, see above).

(D4) Trace map. Let X be a smooth projective variety over k. The trace map $Tr_X : H^{2\dim X}(X)(\dim X) \to k$ is the unique map such that

$$Tr_X(cu_X) = Tr(c)$$

where $c \in H^0(X, \mathcal{O}_X) = H^0_{dR}(X)$ and $Tr(c)$ is the trace for the separable field extension $k \subset H^0(X, \mathcal{O}_X)$.

Remarks. (i) By the remark following our discussion of Poincaré duality we have Poincaré duality with this trace map.
(ii) As usual the structure of Poincaré duality on a contravariant cohomology theory allows us to define pushforwards formally as the adjoint of pullback. More precisely, if $f : X \to Y$ is a morphism of smooth projective varieties over k we define

$$f_* : H^*_{dR}(X) \longrightarrow H^{*-2r}_{dR}(Y)(-r)$$

where $r = \dim X - \dim Y$ by the rule $Tr_Y(f_*(\alpha) \cup \beta) = Tr_X(\alpha \cup f^*\beta)$ for all $\alpha \in H^i_{dR}(X)$ and $\beta \in H^{\dim X - i}_{dR}(Y)(\dim X)$.

Exercise. Show that if $f : X \to Y$ is a generically finite morphism of smooth projective varieties over k of the same dimension then $f_* \circ f^* = \deg(f)\mathrm{id}_{H^*_{dR}(Y)}$.

Exercise. Show that if $f : X \to Y$ is a finite morphism of smooth projective varieties over k of the same dimension, then the map f_* agrees with the map Θ induced by the trace map on de Rham complexes

$$\Theta^* : f_*\Omega^*_{X/k} \longrightarrow \Omega^*_{Y/k}.$$

Hint: Let $\alpha \in H^i_{dR}(X)$ and $\beta \in H^{2d-i}_{dR}(Y)(d)$ where $d = \dim X = \dim Y$. We have $\Theta(\alpha) \cup \beta = \Theta(\alpha \cup f^*\beta)$. Hence it suffices to show that $\Theta(u_X) = u_Y$. Finish by using that $\Theta \circ f^* = \deg(f)$ and $f^*u_Y = \deg(f)u_X$.

(W6). Suppose that X, Y are smooth quasi-projective varieties over k. Let $x \in X$ and $y \in Y$ be closed points. Then $x \times y \subset X \times Y$ is a finite disjoint union of closed points. Note that $\mathcal{O}_{x\times y} = pr_1^*(\mathcal{O}_x) \otimes_{\mathcal{O}_{X\times Y}} pr_2^*(\mathcal{O}_y)$. In fact this is an equality of derived tensor products, which can be seen by a local calculation. Since the projection maps are flat we conclude that

$$u_{X\times Y} = pr_1^*(u_X) \cup pr_2^*(u_Y).$$

From this axiom (W6) follows when X and Y are projective.

Logarithmic de Rham complex. Let X be a smooth projective variety over k, and let Y be a nonsingular divisor on X. In this case the logarithmic de Rham complex $\Omega^*_{X/k}(\log(Y))$ is defined and sits in a short exact sequence of complexes

$$0 \to \Omega^*_{X/k} \to \Omega^*_{X/k}(\log(Y)) \to i_*(\Omega^*_{Y/k})[-1] \to 0.$$

Here $i : Y \to X$ is the inclusion morphism. Zariski locally on X the subscheme Y is cut out by a single equation f. The complex $\Omega^*_{X/k}(\log(Y))$ is generated by $\mathrm{d}\log(f) = \mathrm{d}f/f$ over $\Omega^*_{X/k}$. The *residue* map to $i_*(\Omega_{Y/k})[1]$ is defined by the rule $Res(\mathrm{d}\log(f) \wedge \omega) = \omega|_Y$. Let $U = X \setminus Y$ and let $j : U \to X$ be the open immersion. There are canonical maps

$$\Omega^*_{X/k}(\log(Y)) \longrightarrow j_*(\Omega^*_{U/k}) \longrightarrow Rj_*(\Omega^*_{U/k}).$$

The second arrow is a quasi-isomorphism because the morphism j is affine and hence $Rj_*(\Omega^p_{U/k}) = j_*(\Omega^p_{U/k})$. The first arrow is a quasi-isomorphism also; this is shown by filtering $j_*(\Omega^*_{U/k})$ by pole order and a local computation. See the following papers for this and much more:

A. Grothendieck, On the De Rham cohomology of algebraic varieties. Publications Mathématiques IHES 29 (1966).
P. Deligne, Equations Différentielles à Points Singuliers Réguliers. Lecture Notes in Math. 163.
N. Katz, Nilpotent connections and the monodromy theorem. Applications of a result of Turrittin. Publ. Math. IHES 39 (1970).

The upshot is that there exists a long exact cohomology sequence $H^{i-1}_{dR}(U) \to H^{i-2}_{dR}(Y) \to H^i_{dR}(X) \to H^i_{dR}(U)$ whose formation is compatible with restriction to opens.

Lemma. Suppose that X is a smooth quasi-projective variety over k and suppose that $Z \subset X$ is a closed subset of codimension p with complement $U = X \setminus Z$. Then
(a) $H^i_{dR}(X) \to H^i_{dR}(U)$ is injective for $0 \le i \le 2p-1$.
(b) if Z is geometrically irreducible over k then $\dim_k Ker(H^{2p}_{dR}(X) \to H^{2p}_{dR}(U)) \le 1$.

We prove this by induction on the dimension of X and then by descending induction on the codimension p. Since the singular locus of Z has higher codimension in X, we may replace X by $X \setminus Sing(Z)$ and assume that Z is smooth over k. Next, we choose a general very ample divisor $Y \subset X$ with $Z \subset Y$. A simple Bertini type argument – using that Z is nonsingular – shows that the singular locus of Y has codimension $> p$ in X. Hence we may reduce to the case where we have $Z \subset Y \subset X$ with Z and Y smooth closed subschemes of X and Y of codimension 1. Now we use the following exact diagram

$$\begin{array}{ccccccccc}
H^{i-1}_{dR}(U) & \longrightarrow & H^{i-2}_{dR}(Y) & \longrightarrow & H^i_{dR}(X) & \longrightarrow & H^i_{dR}(U) & \longrightarrow & H^{i+1}_{dR}(Y) \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow \\
H^{i-1}_{dR}(U) & \longrightarrow & H^{i-2}_{dR}(Y \setminus Z) & \longrightarrow & H^i_{dR}(X \setminus Z) & \longrightarrow & H^i_{dR}(U) & \longrightarrow & H^{i+1}_{dR}(Y \setminus Z)
\end{array}$$

The result (a) for $Z \subset X$ follows from the result for $Z \subset Y$. The result (b) is reduced to the case where $Z \subset X$ is a divisor, and then follows from the fact that $H^0_{dR}(Z) = k$ in the geometrically irreducible case.

Chern character of the structure sheaf of a subvariety. Let X be a smooth quasi-projective variety over k. Suppose that $Z \subset X$ is a closed subvariety of codimension p. As usual we think of $\mathcal{O}_Z$ as a coherent sheaf on X. We claim that $ch^{dR}_i(\mathcal{O}_Z) = 0$ for $i = 0, \ldots, p-1$. Namely, the open immersion $j : X \setminus Z \hookrightarrow X$ is flat and hence $j^* ch^{dR}(\mathcal{O}_Z) = ch^{dR}(j^*\mathcal{O}_Z) = ch^{dR}(0) = 0$. Combined with the lemma above this gives the claim.

More generally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that the codimension of the support of $\mathcal{F}$ is $\ge p$. Then $ch^{dR}_i(\mathcal{F}) = 0$ for $i = 0, \ldots, p-1$. This is shown in exactly the same way as above.

Definition (D5) We define the cohomology class of an irreducible subvariety $Z \subset X$ of codimension p to be the element

$$cl^{dR}(Z) := ch_p(\mathcal{O}_Z) \in H^{2p}_{dR}(X)(p).$$

We extend this linearly to general cycles on X. Note that with this definition we have $cl^{dR}(x) = \deg_k(x) u_X$ for a closed point x of a projective X.

Exercise. Suppose that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module with $\dim(Supp(\mathcal{F})) \le d$. Write $d + p = \dim X$. Recall that we associated a d-cycle $[\mathcal{F}]_d$ to $\mathcal{F}$. Show that $cl^{dR}([\mathcal{F}]_d) = ch^{dR}_p(\mathcal{F})$.

Exercise. Show that the cohomology class of $\mathbf{P}^r \subset \mathbf{P}^n$ is equal to $\xi^{n-r} \in H^{2n-2r}_{dR}(\mathbf{P}^n)(n-r)$.

Exercise. Show that if $Y \subset X$ is a codimension 1 subvariety of a smooth variety over k, then $cl^{dR}(Y) = c^{dR}_1(\mathcal{O}_X(Y))$.

Compatibility of cupproduct and intersection product via cycle classes. Suppose that V, W are closed subvarieties of the smooth quasi-projective variety X. Assume that V and W intersect properly. Write $V \cdot W = \sum n_i[Z_i]$ as in the note on intersection theory. Then we claim that

$$cl^{dR}(V) \cup cl^{dR}(W) = \sum_i n_i cl^{dR}(Z_i).$$

This follows from the exercise above, the Tor formula for the intersection product (see note on intersection theory), and the multiplicativity of ch^{dR} under (derived) tensor products.

Exercise: Axiom (W7). Compatibility of cycle classes and exterior product. Suppose given smooth quasi-projective varieties X, Y over k, and closed sub varieties $V \subset X$, $W \subset Y$ Show that

$$\mathcal{O}_{V\times W} = pr_1^*(\mathcal{O}_V) \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} pr_2^*(\mathcal{O}_V).$$

and deduce that $pr_1^* cl^{dR}(V) \cup pr_2^* cl^{dR}(W) = cl^{dR}(V \times W)$.

Rational equivalence and cycle classes. Let X be a smooth quasi-projective variety over k. Let $W \subset X \times \mathbf{P}^1$ be a closed subvariety of dimension $d+1$ dominating $\mathbf{P}^1$. Let W_0, and W_∞ be the scheme theoretic fibres of $W \to \mathbf{P}^1$ over $0, \infty \in \mathbf{P}^1$. Note that the skyscraper sheaves $\kappa(0)$ and $\kappa(\infty)$ are coherent sheaves on $\mathbf{P}^1$ which represent the same element in $K_0(\mathbf{P}^1)$. Hence, by flat pullback $K_0(\mathbf{P}^1) \to K_0(X \times \mathbf{P}^1)$ we observe that $\mathcal{O}_{X\times 0}$ and $\mathcal{O}_{X\times\infty}$ have the same class in $K_0(X \times \mathbf{P}^1)$. Hence we deduce that the structure sheaves of the scheme theoretic intersections

$$\mathcal{O}_{W\cap X\times 0} = \mathcal{O}_W \otimes_{\mathcal{O}_{X\times\mathbf{P}^1}} \mathcal{O}_{X\times 0} = \mathcal{O}_W \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times\mathbf{P}^1}} \mathcal{O}_{X\times 0}$$

and $\mathcal{O}_{W\cap X\times\infty}$ have the same class in $K_0(X \times \mathbf{P}^1)$. Using the pushforward homomorphism $pr_{X,*} : K_0(X \times \mathbf{P}^1) \to K_0(X)$ we deduce that

$$\mathcal{O}_{W_0} = pr_{X,*}(\mathcal{O}_{W\cap X\times 0}) = Rpr_{X,*}(\mathcal{O}_{W\cap X\times 0})$$

and $\mathcal{O}_{W_\infty}$ have the same class in $K_0(X)$. We conclude that $cl^{dR}(W_0) = cl^{dR}(W_\infty)$ in the algebraic de Rham cohomology of X. In other words, the cycle class map into algebraic de Rham cohomology factors through rational equivalence.

Remark. At this point we basically have all the axioms of a Weil cohomology theory except for (W8), which is the compatibility of cycle classes with pushforward. Namely, (W10) is obvious and (W9) will follow if we can prove (W8) because pullback of cycles under a map $f : X \to Y$ of smooth projective varieties is defined via the rule $f^*\alpha = pr_{X,*}(\Gamma_f \cdot pr_Y^*\alpha)$ and we already have compatibility of cycle classes with rational equivalence, (proper) intersection products and flat pullback.

Lemma. Suppose that for every morphism of smooth projective varieties $f : X \to Y$ we have $f_*(1) = cl^{dR}(f_*[X])$. Then cycle classes are compatible with pushforward, i.e., axiom (W8) holds.

Let $f : X \to Y$ be a morphism of smooth projective varieties over k. Let $Z \subset X$ be a closed subvariety. Let e be the degree of the map $Z \to f(Z)$. Consider a nonsingular projective alteration $Z' \to Z$. Say the degree of $Z' \to Z$ is d. Hence the degree of $Z' \to f(Z)$ is de. We get a commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{g} & X \\ & \searrow^{h} & \downarrow{\scriptstyle f} \\ & & Y \end{array}.$$

The assumption of the lemma says that $g_*(1) = d\ cl^{dR}(Z)$ and $h_*(1) = de\ cl^{dR}(f(Z))$. Hence

$$f_*(cl^{dR}(Z)) = (1/d)f_*(g_*(1)) = (1/d)h_*(1) = (1/d)ed\ cl^{dR}(f(Z)) = e\ cl^{dR}(f(Z))$$

as desired.

Lemma. Suppose that for every closed immersion of smooth projective varieties $i : X \to Y$ we have $i_*(1) = cl^{dR}(f_*[X])$. Then cycle classes are compatible with pushforward, i.e., axiom (W8) holds.

Proof. Let $f : X \to Y$ be a morphism of smooth projective varieties over k. Choose a closed immersion $X \to \mathbf{P}^n$ with $n >> 0$. This gives a factorization $X \to Y \times \mathbf{P}^n \to Y$ of the morphism f. Hence in order to prove the lemma (by functoriality of pushforward) it suffices to prove that pushforward along $pr_Y : Y \times \mathbf{P}^n \to Y$ commutes with cycle classes. For this we use the ring isomorphism

$$A^*(Y \times \mathbf{P}^n) = A^*(Y) \otimes_{\mathbf{Q}} A^*(\mathbf{P}^n)$$

of $\mathbf{Q}$-Chow rings modulo rational equivalence from intersection theory. (It is a very special case of the projective space bundle formula.) Since *both* pushforward on cohomology and on cycles satisfies the projection formula it suffices to check that $pr_{Y,*}(cl^{dR}(Y \times \mathbf{P}^r)) = 0$ for $0 < r \leq n$ and $= 1$ for $r = 0$. Left as an exercise. *EndProof.*

Pushforward along an effective very ample divisor. Let $Y \subset X$ be a smooth divisor on the smooth projective variety X over k. Let $i : Y \to X$ denote the closed immersion. Assume that Y is a *very ample* divisor. Then $i_*(1) = cl^{dR}(Y)$.

Because Y is very ample there exists a finite morphism $f : X \to \mathbf{P}^n$ with $n = \dim X$ such that $Y = f^{-1}(\mathbf{P}^{n-1})$ scheme theoretically. We get the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbf{P}^{n-1} & \xrightarrow{i'} & \mathbf{P}^n. \end{array}$$

We claim that $(f')_* \circ i^* = (i')^* \circ f_*$ on algebraic de Rham cohomology. Perhaps the easiest way of seeing this is to consider the trace maps on the de Rham complexes. These are the vertical maps in the following diagram

$$\begin{array}{ccc} f_*\Omega^*_{X/k} & \longrightarrow & (i')_*(f')_*\Omega^*_{Y/k} \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta} \\ \Omega^*_{\mathbf{P}^n/k} & \longrightarrow & (i')_*\Omega^*_{\mathbf{P}^{n-1}/k}. \end{array}$$

and the horizontal maps correspond to the pullback maps. Using that the maps induced by Θ are equal to the pushforward on cohomology, the claim follows by verifying that the diagram of maps of complexes is commutative. This (local) verification is left to the reader.

We deduce from $(f')_* \circ i^* = (i')^* \circ f_*$ formally that $i_*(1) = f^*(i')_*(1)$. Since also $f^*cl^{dR}(\mathbf{P}^{n-1}) = cl^{dR}(Y)$ by flatness of f, we see that it suffices to prove $(i')_*(1) = cl^{dR}(\mathbf{P}^{n-1})$. This is clear from the computation of cohomology of $\mathbf{P}^n$ above.

Pushforward along an effective divisor. Let $Y \subset X$ be a smooth divisor on the smooth projective variety X over k. Let $i : Y \to X$ denote the closed immersion. Then $i_*(1) = cl^{dR}(Y)$.

Choose a very ample divisor $Y' \subset X$ such that $Y \cup Y'$ is very ample as well. We may choose Y' such that Y' is smooth and $Y \cap Y'$ is smooth of codimension 2 in X. Now we argue as above using a finite morphism $f : X \to \mathbf{P}^n$ such that $f^{-1}(\mathbf{P}^{n-1}) = Y \cup Y'$ scheme theoretically. We get the following commutative diagram

$$\begin{array}{ccc} Y \coprod Y' & \xrightarrow{i,i'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbf{P}^{n-1} & \xrightarrow{i''} & \mathbf{P}^n. \end{array}$$

We claim that $(f')_* \circ (i^* \oplus (i')^*) = (i'')^* \circ f_*$ on algebraic de Rham cohomology. Perhaps the easiest way of seeing this is to consider the trace maps on the de Rham complexes. These are the vertical maps in the following diagram

$$\begin{array}{ccc} f_*\Omega^*_{X/k} & \longrightarrow & (i'')_*(f')_*\Omega^*_{Y/k} \oplus (i'')_*(f')_*\Omega^*_{Y'/k} \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta} \\ \Omega^*_{\mathbf{P}^n/k} & \longrightarrow & (i'')_*\Omega^*_{\mathbf{P}^{n-1}/k}. \end{array}$$

and the horizontal maps correspond to the pullback maps. Using that the maps induced by Θ are equal to the pushforward on cohomology, the claim follows by verifying that the diagram of maps of complexes is commutative. This (local) verification is left to the reader.

Because $(f')^*(1) = 1 \oplus 1$ we deduce from $(f')_* \circ (i^* \oplus (i')^*) = (i'')^* \circ f_*$ formally that $i_*(1) + (i')_*(1) = f^*(i'')_*(1)$. Since also $f^* cl^{dR}(\mathbf{P}^{n-1}) = cl^{dR}(Y) + cl^{dR}(Y')$ by flatness of f, and since $(i')_*(1) = cl^{dR}(\mathbf{P}^{n-1})$ we obtain $i_*(1) + (i')_*(1) = cl^{dR}(Y) + cl^{dR}(Y')$. Now the result follows from the fact that we already proved $(i')_*(1) = cl^{dR}(Y')$ above (because Y' is very ample).

Pushforward along a global complete intersection morphism. Suppose that X is a smooth projective variety over k. Let $H_1, \ldots, H_c$ be effective divisors on X such that for every $1 \le i \le c$ the intersection $H_1 \cap \ldots \cap H_i$ is a smooth variety. Set $Y = H_1 \cap \ldots \cap H_c$, and denote $i : Y \to X$ the closed immersion. We claim that $i_*(1) = cl^{dR}(Y)$ in $H^{2c}_{dR}(X)(c)$.

Proof. We do this by induction on c; the case $c = 1$ is done above. Assume $c > 1$. Set $X' = H_1$ and denote $i' : Y \to X'$ and $i'' : X' \to X$ the closed immersions. Also denote $H'_2, \ldots, H'_c$ the intersections $X' \cap H_2, \ldots, X' \cap H_c$. By induction on the codimension we have $(i')_*(1) = cl^{dR}_{X'}(Y)$ in the de Rham cohomology of X', and $(i'')_*(1) = cl^{dR}_X(X')$ in the de Rham cohomology of X'. Since we have see that taking cohomology classes is compatible with intersection products we obtain that $cl^{dR}_{X'}(Y) = cl_{X'}(H'_2) \cup \ldots \cup cl_{X'}(H'_c)$. By an exercise above we have $cl_{X'}(H'_i) = c_1^{dR}(\mathcal{O}_{X'}(H'_i))$. Since Chern classes are functorial for pullbacks we deduce $cl_{X'}(H'_i) = c_1^{dR}(\mathcal{O}_{X'}(H'_i)) = (i'')^* c_1^{dR}(\mathcal{O}_X(H_i)) = (i'')^* cl_X(H_i)$. Therefore,

$$
\begin{aligned}
i_*(1) &= (i'')_*(i')_*(1) \\
&= (i'')_* cl^{dR}_{X'}(Y) \\
&= (i'')_* cl_{X'}(H'_2) \cup \ldots \cup cl_{X'}(H'_c) \\
&= (i'')_*(i'')^* \Big(cl_X(H_2) \cup \ldots \cup cl_X(H_c) \Big) \\
&= (i'')_*(1) \cup cl_X(H_2) \cup \ldots \cup cl_X(H_c) \\
&= cl_X(H_1) \cup cl_X(H_2) \cup \ldots \cup cl_X(H_c) \\
&= cl_X(Y).
\end{aligned}
$$

EndProof.

Pushforward along a general closed immersion. Let $i : Y \to X$ be a closed immersion of smooth projective varieties over k. Say the codimension is $c > 1$. Denote $b : \tilde{X} \to X$ the blow up of X in Y, and denote E the exceptional divisor. We have the usual commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{i'} & \tilde{X} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle b} \\
Y & \xrightarrow{i} & X
\end{array}
$$

Let $H_1, H_2, \ldots, H_{c-1}$ be very ample divisors on $\tilde{X}$. By Bertini, we may choose these such that the intersection $Y' = E \cap H_1 \cap \ldots \cap H_{c-1}$ is a smooth variety of codimension $c - 1$ in E. Note that we may also assume that the induced morphism $g : Y' \to Y$ is finite. So now the diagram looks like

$$
\begin{array}{ccccc}
Y' & \xrightarrow{i''} & E & \xrightarrow{i'} & \tilde{X} \\
 & \searrow{\scriptstyle g} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle b} \\
 & & Y & \xrightarrow{i} & X
\end{array}
$$

Since g is finite, say of degree d we have that $g_*(1) = d \cdot 1$. Thus we see that

$$i_*(1) = (1/d) g_* i_*(1) = (1/d) f_*(i')_*(i'')_*(1) = (1/d) f_* cl^{dR}(Y').$$

The last step because Y' is a global complete intersection in $\tilde{X}$. We show below that $(f_* cl^{dR}(Y'))|_U = cl^{dR}(Y')|_U$. (This is not trivial with our definition of f_*!) Since cycle classes are compatible with flat pullbacks we conclude that $cl^{dR}(Y')|_U = 0$ and hence we conclude that $i_*(1)|_U = (1/d)(f_* cl^{dR}(Y'))|_U = (1/d)cl^{dR}(Y')|_U = 0$. The seemingly innocuous statement $i_*(1)|_U = 0$ is the key to proving axiom (W8).

Assume that Y is geometrically irreducible over k. In that case we now know that both $i_*(1)$ and $cl^{dR}(Y)$ lie in the 1-dimensional subspace

$$\mathrm{Ker}(H^{2p}_{dR}(X)(p) \longrightarrow H^{2p}_{dR}(X \setminus Y))$$

(see earlier result). Hence it suffices to show that $Tr_X(i_*(1) \cup \beta) = Tr_X(cl^{dR}(Y) \cup \beta)$ for one class β such that the cup products are nonzero. Take $\beta = cl^{dR}(H)^{\dim Y}$, where H is a very ample divisor on X. We have enough theory at our disposal to see that

$$Tr_X(i_*(1) \cup cl^{dR}(H)^{\dim Y}) = \deg_H(Y) = Tr_X(cl^{dR}(Y) \cup cl^{dR}(H)^{\dim Y})$$

in this case. (The RHS because classes are compatible with intersection products and the LHS because classes of divisors are compatible with pullbacks – via chern classes.) This finishes the proof in the case that Y is geometrically irreducible.

The general case follows from the geometrically irreducible case by doing a base field extension. This is true but it is a bit annoying to write out completely†. Namely, one has to show that for a variety X over k, and a finite field extension $k \subset k'$ one has $H^*_{dR}(X \times_{Spec(k)} Spec(k')) = H^*_{dR}(X) \otimes_k k'$ and moreover that this is compatible with all the constructions we made above. In our situation $Y \subset X$ as above we then choose k' such that $Y \times Spec(k')$ becomes a union of geometrically irreducible varieties. Details left to the reader.

The trace map for a nonsingular blowup. Let $i : Y \to X$ be a closed immersion of smooth projective varieties over k. Say the codimension is $c > 1$. Denote $b : \tilde{X} \to X$ the blow up of X in Y, and denote E the exceptional divisor. Let $U = X \setminus Y = \tilde{X} \setminus E$. We have the usual commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{i'} & \tilde{X} & \xleftarrow{j} & U \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle b} & & \| \\ Y & \xrightarrow{i} & X & \longleftarrow & U \end{array}$$

We want to show that

$$f_*(\alpha)|_U = \alpha|_U$$

for any $\alpha \in H^*_{dR}(\tilde{X})$. This is not a triviality since we have only defined the pushforward map on cohomology for morphisms of smooth projective varieties.

Proof.♡ First we introduce the complex

$$\mathcal{K}^* := \mathrm{Ker}(\Omega^*_{\tilde{X}/k} \to (i')_* \Omega^*_{E/k}).$$

A local calculations shows that $\mathcal{K}^*$ is a complex of abelian groups, whose terms are finite locally free $\mathcal{O}_{\tilde{X}}$-modules and that the cupproduct on $\Omega^*_{\tilde{X}/k}$ extends to a pairing

$$\mathcal{K}^* \times \Omega^*_{\tilde{X}/k}(\log E) \longrightarrow \Omega^*_{\tilde{X}/k}$$

which satisfies the conditions (1)–(5) of the section on duality above. Hence we deduce for $a + b = 2\dim X$ that $\mathbf{H}^b(\tilde{X}, \mathcal{K}^*)$ and $H^a_{dR}(U) = \mathbf{H}^a(X, \Omega^*_{\tilde{X}/k}(\log E))$ are canonically dual. This duality has the property that the maps

$$H^a_{dR}(\tilde{X}) \longrightarrow H^a_{dR}(U)$$

† Is there is a formal argument deducing the general case from the geometrically irreducible one?

♡ This proof is absolutely horrid. It uses the compactly supported de Rham cohomology of U without properly introducing compactly supported cohomology.

and

$$\mathbf{H}^b(\tilde{X}, \mathcal{K}^*) \longrightarrow H^b_{dR}(\tilde{X})$$

are transpose of each other. Below we will write this as follows: for $\alpha \in H^a_{dR}(\tilde{X})$, and $\beta \in \mathbf{H}^b(\tilde{X}, \mathcal{K})$ we have

$$\langle \alpha|_U, \beta \rangle = \langle \alpha, Im(\beta) \rangle$$

where $Im(\beta)$ indicate the image of β in the de Rham cohomology of $\tilde{X}$. For any integer $n > \dim X$ consider the subcomplex $\mathcal{K}(n)^*$ of $\mathcal{K}^*$ with terms $\mathcal{K}(n)^p = \mathcal{I}_E^{n-p}\mathcal{K}^p = \mathcal{O}_{\tilde{X}}((-n+p)E) \otimes \mathcal{K}^p$. Dually consider the complex $\Omega^*_{\tilde{X}/k}([nE])$ with terms $\Omega^p_{\tilde{X}/k}([nE]) = \mathcal{O}((n - \dim X + p)E) \otimes \Omega^p_{\tilde{X}/k}$. Again there is a pairing $\mathcal{K}(n) \times \Omega^*_{\tilde{X}/k}([nE]) \to \Omega^*_{\tilde{X}/k}$ inducing a duality in cohomology compatible with maps as above. Since we know that the first and the last term of

$$\Omega^*_{\tilde{X}/k} \subset \Omega^*_{\tilde{X}/k}([nE]) \subset \Omega^*_{\tilde{X}}(*E) = j_*(\Omega^*_{U/k})$$

are quasi-isomorphic, we know that

$$H^a_{dR}(U) = \mathbf{H}^a(\tilde{X}, \Omega^*_{\tilde{X}/k}(\log E)) \longrightarrow \mathbf{H}^a(\tilde{X}, \Omega^*_{\tilde{X}/k}([nE]))$$

is injective. Dually we deduce that

$$\mathbf{H}^b(\tilde{X}, \mathcal{K}(n)^*) \longrightarrow \mathbf{H}^b(\tilde{X}, \mathcal{K}^*)$$

is surjective. Next, since $\mathcal{I}_E$ is relatively ample for $f : \tilde{X} \to X$ we have for n large enough a canonical quasi-isomorphism

$$f_*\mathcal{K}(n)^* \longrightarrow Rf_*\mathcal{K}(n)^*$$

Note also that $f_*\mathcal{K}(n)^*|_U = \Omega^*_{U/k}$ since after all $\mathcal{K}(n)|_U$ equals $\Omega^*_{U/k}$. The sheaves $f_*\mathcal{K}(n)^i$ are coherent and torsion free, and hence by Hartog's theorem the isomorphism $f_*\mathcal{K}(n)^i|_U = \Omega^i_{U/k}$ extends to a map of complexes

$$f_*\mathcal{K}(n)^* \longrightarrow \Omega^*_{X/k}$$

because the codimension $c \geq 2$. (Still for some large n.) Of course the construction shows this map is compatible with the map $\mathcal{K}(n)^* \to \Omega^*_{\tilde{X}/k}$ via $f^{-1}\Omega^*_{X/k} \to \Omega^*_{\tilde{X}/k}$ (even term by term). Take any $\beta \in \mathbf{H}^b(\tilde{X}, \mathcal{K}^*)$. Lift it to some $\beta(n) \in \mathbf{H}^b(\tilde{X}, \mathcal{K}(n)^*)$. Write $\beta(n)$ as the image of $\beta(n)' \in \mathbf{H}^b(X, f_*\mathcal{K}(n)^*)$ (possible by the quasi-isomorphism above). Let $\beta' \in H^b_{dR}(X)$ be the image of $\beta(n)'$. Then $f^*\beta'$ equals the image $Im(\beta)$ of β in $H^b_{dR}(\tilde{X})$ by the compatibility of maps mentioned above. Using this we may compute

$$\begin{aligned}
\langle f_*(\alpha)|_U, \beta \rangle &= \langle f^* f_*(\alpha)|_U, \beta \rangle \\
&= \langle f^* f_*(\alpha), Im(\beta) \rangle \\
&= \langle f^* f_*(\alpha), f^*\beta' \rangle \\
&= \langle f_*(\alpha), \beta' \rangle \\
&= \langle \alpha, f^*\beta' \rangle \\
&= \langle \alpha, Im(\beta) \rangle \\
&= \langle \alpha|_U, \beta \rangle
\end{aligned}$$

The first equality because pullback is functorial! The second equality we saw above. The third equality by our choice of β'. The fourth equality because $f^*u_X = u_{\tilde{X}}$. The fifth equality because of the definition of f_*. The sixth equality because of our choice of β'. The seventh equality we saw above. Ok, and since this is true for every β we get by the duality statement above that $f_*(\alpha)|_U = \alpha|_U$. *EndProof.*

Appendix on cupproduct.

In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on X there is a cupproduct map $H^i(X,\mathcal{F}) \times H^j(X,\mathcal{G}) \to H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G})$. The easiest way I know how to define it is to compute cohomology using Cech cocyles and write out the formula for the cup product. See below. If you are worried about the fact that cohomology may not equal Cech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any site.

Let $\mathcal{F}^*$ be a bounded below complex of sheaves of abelian groups on X. We can (often) compute $\mathbf{H}^n(X, \mathcal{F}^*)$ using Cech cocycles. Namely, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. Consider the (simple) complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$ with degree n term

$$s^n\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) = \prod\nolimits_{p+q=n} \mathcal{F}^q(U_{i_0 \ldots i_p})$$

with a typical element denoted $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ so that $\alpha_{i_0 \ldots i_p} \in \mathcal{F}^q(U_{i_0 \ldots i_p})$, in other words the $\mathcal{F}$-degree of $\alpha_{i_0 \ldots i_p}$ is q. We indicate this by the formula $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_p}) = q$. The differential of an element α of degree n is

$$\begin{aligned} d(\alpha)_{i_0 \ldots i_{p+1}} &= d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+1}}) + (-1)^{n-p} \sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \\ &= d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+1}}) + \sum\nolimits_{j=0}^{p+1} (-1)^{j+n-p} \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \end{aligned}$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}$. An expression such as $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$ means the restriction of $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \in \mathcal{F}(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})$ to $U_{i_0 \ldots i_{p+1}}$. To check this is a complex, let α be an element of degree n in $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$, so that $d(\alpha)$ has degree $n+1$. We compute:

$$\begin{aligned} d^2(\alpha)_{i_0 \ldots i_{p+2}} &= d_{\mathcal{F}}(d(\alpha)_{i_0 \ldots i_{p+2}}) + (-1)^{(n+1)-(p+1)} \sum\nolimits_{j=0}^{p+2} (-1)^j d(\alpha)_{i_0 \ldots \hat{i}_j \ldots i_{p+2}} \\ &= d_{\mathcal{F}}(d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+2}})) \\ &+ d_{\mathcal{F}}\Big((-1)^{n-(p+1)} \sum\nolimits_{j=0}^{p+2} (-1)^j \ \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+2}}\Big) \\ &+ (-1)^{(n+1)-(p+1)} \sum\nolimits_{j=0}^{p+2} (-1)^j d_{\mathcal{F}}(\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+2}}) \\ &+ (-1)^{(n+1)-(p+1)} \sum\nolimits_{j=0}^{p+2} (-1)^{j+n-p} \sum\nolimits_{j'=0 \ldots j-1} (-1)^{j'} \alpha_{i_0 \ldots \hat{i}_{j'} \ldots \hat{i}_j \ldots i_{p+1}} \\ &+ (-1)^{(n+1)-(p+1)} \sum\nolimits_{j=0}^{p+2} (-1)^{j+n-p} \sum\nolimits_{j'=j+1 \ldots p+2} (-1)^{j'-1} \alpha_{i_0 \ldots \hat{i}_j \ldots \hat{i}_{j'} \ldots i_{p+1}} \end{aligned}$$

which equals zero by the nullity of $d^2_{\mathcal{F}}$, a trivial sign change between the second and third terms, and the usual argument for the last two double Cech terms.

The construction of $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$ is functorial in $\mathcal{F}^*$. As well there is a functorial transformation

$$\Gamma(X, \mathcal{F}^*) \longrightarrow s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{I_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

$$T_t : s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) \longrightarrow s\mathcal{C}^*(\mathcal{V}, \mathcal{F}^*).$$

This is defined by the rule

$$T_t(\alpha)_{j_0 \ldots j_p} = \alpha_{t(j_0) \ldots t(j_p)}|_{V_{j_0 \cdots j_p}}$$

Given two maps $t, t' : J \to I$ as above the maps T_t and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0 \ldots j_p} = (-1)^{n+p} \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}$$

for an element α of degree n. This works because of the following computation, again with α an elemement of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$\begin{aligned}
&(d(h(\alpha)) + h(d(\alpha)))_{j_0\ldots j_p}\\
&= d_{\mathcal{F}}(h(\alpha)_{j_0\ldots j_p}) + (-1)^{(n-1)-(p-1)} \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{j_0\ldots \hat{j}_k \ldots j_p}\\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^a d(\alpha)_{t(j_0)\ldots t(j_a) t'(j_a)\ldots t'(j_p)}\\
&= (-1)^{n+p} \sum\nolimits_{a=0}^{p} (-1)^a d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)})\\
&+ (-1)^{(n-1)-(p-1)} \sum\nolimits_{k=0}^{p} (-1)^{k+n+p-1} \sum\nolimits_{a=0}^{k-1} (-1)^a \alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)}\\
&+ (-1)^{(n-1)-(p-1)} \sum\nolimits_{k=0}^{p} (-1)^{k+n+p-1} \sum\nolimits_{a=k+1}^{p} (-1)^{a-1} \alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}\\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^a d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)})\\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^{a+n-p} \sum\nolimits_{k=0}^{a} (-1)^k \alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}\\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^{a+n-p} \sum\nolimits_{k=a}^{p} (-1)^{k+1} \alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)}\\
&= \alpha_{t(j_0)\ldots t(j_p)} - \alpha_{t'(j_0)\ldots t'(j_p)} = T_t(\alpha)_{j_0\ldots j_p} - T_{t'}(\alpha)_{j_0\ldots j_p}
\end{aligned}$$

We leave it to the reader to verify the cancellations. It follows that the induced map

$$H^*(T_t) : H^*(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)) \to H^*(s\mathcal{C}^*(\mathcal{V}, \mathcal{F}^*))$$

is independend of the choice of t. We define Cech hypercohomology as the limit of the Cech cohomology groups over all refinements via the maps $H^*(T_t)$.

Let $\mathcal{I}^*$ be a bounded below complex of injectives. Consider the map $\Gamma(X, \mathcal{I}^*) \to s\mathcal{C}^*(\mathcal{U}, \mathcal{I}^*)$ defined in degree n by $i \mapsto \alpha = \{\alpha_{i_0\ldots i_p}\}$ with $\alpha_{i_0} = i|_{U_{i_0}}$ and $\alpha_{i_0\ldots i_p} = 0$ for $p > 0$. This is a quasi-isomorphism of complexes of abelian groups (prove by a spectral sequence argument on the double complex $\mathcal{C}^*(\mathcal{U}, \mathcal{I}^*)$). Suppose $\mathcal{F}^* \to \mathcal{I}^*$ is a quasi-isomorphism of $\mathcal{F}^*$ into a bounded below complex of injectives. The hypercohomology $\mathbf{H}^*(X, \mathcal{F}^*)$ is defined to be $H^*(\Gamma(X, \mathcal{I}^*))$. Thus the corresponding map $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) \to s\mathcal{C}^*(\mathcal{U}, \mathcal{I}^*)$ induces maps $H^*(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)) \to \mathbf{H}^*(X, \mathcal{F}^*)$. In the limit this induces a map of Cech hypercohomology into the cohomology, which is usually an isomorphism and is always an isomorphism if we use hypercoverings.

Consider the map $\tau : s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) \to s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$ defined by

$$\tau(\alpha)_{i_0\ldots i_p} = (-1)^{p(p+1)/2} \alpha_{i_p\ldots i_0}.$$

Then we have for an element α of degree n that

$$\begin{aligned}
d(\tau(\alpha))_{i_0\ldots i_{p+1}} &= d_{\mathcal{F}}(\tau(\alpha)_{i_0\ldots i_{p+1}}) + (-1)^{n-p} \sum\nolimits_{j=0}^{p+1} (-1)^j \tau(\alpha)_{i_0\ldots \hat{i}_j \ldots i_{p+1}}\\
&= (-1)^{(p+1)(p+2)/2} d_{\mathcal{F}}(\alpha_{i_{p+1}\ldots i_0})) + (-1)^{n-p} \sum\nolimits_{j=0}^{p+1} (-1)^{j+p(p+1)/2} \alpha_{i_{p+1}\ldots \hat{i}_j \ldots i_0}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\tau(d(\alpha))_{i_0\ldots i_{p+1}} &= (-1)^{(p+1)(p+2)/2} d(\alpha)_{i_{p+1}\ldots i_0}\\
&= (-1)^{(p+1)(p+2)/2} d_{\mathcal{F}}(\alpha_{i_{p+1}\ldots i_0}) + (-1)^{(p+1)(p+2)/2+n-p} \sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_{p+1}\ldots \hat{i}_{p+1-j}\ldots i_0}\\
&= (-1)^{(p+1)(p+2)/2} d_{\mathcal{F}}(\alpha_{i_{p+1}\ldots i_0}) + (-1)^{(p+1)(p+2)/2+n-p} \sum\nolimits_{j=0}^{p+1} (-1)^{j-p-1} \alpha_{i_{p+1}\ldots \hat{i}_j \ldots i_0}
\end{aligned}$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p + 1 \bmod 2$. In other words τ is an endomorphism of the complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$. Note that the diagram

$$\begin{array}{ccc}
\Gamma(X, \mathcal{F}^*) & \longrightarrow & s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)\\
\downarrow \mathrm{id} & & \downarrow \tau\\
\Gamma(X, \mathcal{F}^*) & \longrightarrow & s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)
\end{array}$$

commutes. In addition τ is clearly compatible with refinements. This proves that τ acts as the identity on Cech hypercohomology (i.e., in the limit – provided Cech hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). To see this use a quasi-isomorphism $\mathcal{F}^* \to \mathcal{I}^*$ of $\mathcal{F}^*$ into a bounded below complex of injectives as before. We claim that τ actually is homotopic to the identity on the simple Cech complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$. To prove this, we use as homotopy

$$h(\alpha)_{i_0 \ldots i_p} = (-1)^{n+p} \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{i_0 \ldots i_a i_{p-a} \ldots i_0}$$

for α of degree n. As usual we omit writing $|_{U_{i_0 \ldots i_p}}$. This works because of the following computation, again with α an elemement of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$\begin{aligned}
&(d(h(\alpha)) + h(d(\alpha)))_{i_0 \ldots i_p} \\
&= d_{\mathcal{F}}(h(\alpha)_{i_0 \ldots i_p}) + (-1)^{(n-1)-(p-1)} \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{i_0 \ldots \hat{i}_k \ldots i_p} \\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^a d(\alpha)_{i_0 \ldots i_a i_{p-a} \ldots i_0} \\
&= (-1)^{n+p} \sum\nolimits_{a=0}^{p} (-1)^a d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_{p-a} \ldots i_0}) \\
&+ (-1)^{(n-1)-(p-1)} \sum\nolimits_{k=0}^{p} (-1)^{k+n+p-1} \sum\nolimits_{a=0}^{k-1} (-1)^a \alpha_{i_0 \ldots i_a i_{p-a} \ldots \hat{i_{p-k}} \ldots i_0} \\
&+ (-1)^{(n-1)-(p-1)} \sum\nolimits_{k=0}^{p} (-1)^{k+n+p-1} \sum\nolimits_{a=k+1}^{p} (-1)^{a-1} \alpha_{i_0 \ldots \hat{i}_k \ldots i_a i_{p-a} \ldots i_0} \\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^a d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_{p-a} \ldots i_0}) \\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^{a+n-p} \sum\nolimits_{k=0}^{a} (-1)^k \alpha_{i_0 \ldots \hat{i}_k \ldots i_a i_{p-a} \ldots i_0} \\
&+ (-1)^{n+1+p} \sum\nolimits_{a=0}^{p} (-1)^{a+n-p} \sum\nolimits_{k=a}^{p} (-1)^{k+1} \alpha_{i_0 \ldots i_a i_{p-a} \ldots \hat{i_{p-k}} \ldots i_0} \\
&= \alpha_{i_0 \ldots i_p} - \alpha_{i_p \ldots i_0}
\end{aligned}$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded complexes complexes of sheaves $\mathcal{F}^*$ and $\mathcal{G}^*$. We define the complex $Tot(\mathcal{F}^* \otimes_{\mathbf{Z}} \mathcal{G}^*)$ to be to complex with terms $\otimes_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$ and differential according to the rule $d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta)$ when α and β are homogenous. We apply the same rule to define the total complex associated to a tensor product of complexes of abelian groups (the case when the space is a point).

Suppose that M^* and N^* are two bounded below complexes of abelian groups. Then if m, resp. n is a cocycle for M^*, resp. N^*, it is immediate that $m \otimes n$ is a cocycle for $Tot(M^* \otimes N^*)$. Hence a cupproduct

$$H^i(M^*) \times H^j(N^*) \to H^{i+j}(Tot(M^* \otimes N^*)).$$

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

$$Tot(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) \otimes_{\mathbf{Z}} s\mathcal{C}^*(\mathcal{U}, \mathcal{G}^*)) \longrightarrow s\mathcal{C}^*(\mathcal{U}, Tot(\mathcal{F}^* \otimes \mathcal{G}^*)),\ \alpha \otimes \beta \mapsto \alpha \cup \beta.$$

This is done by the rule

$$(\alpha \cup \beta)_{i_0 \ldots i_p} = \sum\nolimits_{r=0}^{p} (-1)^{r(m-(p-r))} \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_p}.$$

where α has degree n and β has degree m. Note that $\alpha \cup \beta$ has degree $n+m$. For an explanation of the sign see the paper "Higher order operations in Deligne cohomology" by Denninger who refers to the paper "cohomologie a support propres" by Deligne for a more precise explanation. To check this is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup d(\beta)$$

because $d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta)$ is the formula for the differential on $Tot(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) \otimes_{\mathbf{Z}} s\mathcal{C}^*(\mathcal{U}, \mathcal{G}^*))$. We compute first

$$\begin{aligned} d(\alpha \cup \beta)_{i_0 \ldots i_{p+1}} &= d_{\mathcal{F} \otimes \mathcal{G}}\Big((\alpha \cup \beta)_{i_0 \ldots i_{p+1}}\Big) + (-1)^{n+m-p} \sum\nolimits_{j=0}^{p+1} (-1)^j (\alpha \cup \beta)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \\ &= \sum\nolimits_{r=0}^{p+1} (-1)^{r(m-(p+1-r))} d_{\mathcal{F} \otimes \mathcal{G}}(\alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}}) \\ &+ (-1)^{n+m-p} \sum\nolimits_{j=0}^{p+1} (-1)^j \sum\nolimits_{r=0}^{j-1} (-1)^{r(m-(p-r))} \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} \\ &+ (-1)^{n+m-p} \sum\nolimits_{j=0}^{p+1} (-1)^j \sum\nolimits_{r=j+1}^{p+1} (-1)^{(r-1)(m-(p+1-r))} \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \end{aligned}$$

On the other hand

$$\begin{aligned} (d(\alpha) \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum_{r=0}^{p+1} (-1)^{r(m-(p+1-r))} d(\alpha)_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \\ &= \sum_{r=0}^{p+1} (-1)^{r(m-(p+1-r))} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} \\ &+ \sum_{r=0}^{p+1} (-1)^{r(m-(p+1-r))+n-(r-1)} \sum\nolimits_{j=0}^{r} (-1)^j \alpha_{i_0 \ldots \hat{i_j} \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \end{aligned}$$

and

$$\begin{aligned} (-1)^n (\alpha \cup d(\beta))_{i_0 \ldots i_{p+1}} &= (-1)^n \sum_{r=0}^{p+1} (-1)^{r(m+1-(p+1-r))} \alpha_{i_0 \ldots i_r} \otimes d(\beta)_{i_r \ldots i_{p+1}} \\ &= (-1)^n \sum_{r=0}^{p+1} (-1)^{r(m+1-(p+1-r))} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}}) \\ &+ (-1)^n \sum_{r=0}^{p+1} (-1)^{r(m+1-(p+1-r))+m-(p-r)} \sum\nolimits_{j=r}^{p+1} (-1)^{j-r} \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i_j} \ldots i_{p+1}} \end{aligned}$$

Now you can see the desired equality.

Associativity of the cupproduct. Suppose that $\mathcal{F}^*$, $\mathcal{G}^*$ and $\mathcal{H}^*$ are bounded below complexes of abelian groups on X. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$Tot(Tot(\mathcal{F}^* \otimes_{\mathbf{Z}} \mathcal{G}^*) \otimes_{\mathbf{Z}} \mathcal{H}^*) \longrightarrow Tot(\mathcal{F}^* \otimes_{\mathbf{Z}} Tot(\mathcal{G}^* \otimes_{\mathbf{Z}} \mathcal{H}^*)).$$

Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if α has degree a, β has degree b and γ has degree c, then

$$\begin{aligned} ((\alpha \cup \beta) \cup \gamma)_{i_0 \ldots i_p} &= \sum_{r=0}^{p} (-1)^{r(c-(p-r))} (\alpha \cup \beta)_{i_0 \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \\ &= \sum_{r=0}^{p} (-1)^{r(c-(p-r))} \sum_{s=0}^{r} (-1)^{s(b-(r-s))} \alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \end{aligned}$$

and

$$\begin{aligned} (\alpha \cup (\beta \cup \gamma)_{i_0 \ldots i_p} &= \sum_{s=0}^{p} (-1)^{s(b+c-(p-s))} \alpha_{i_0 \ldots i_s} \otimes (\beta \cup \gamma)_{i_s \ldots i_p} \\ &= \sum_{s=0}^{p} (-1)^{s(b+c-(p-s))} \sum_{r=s}^{p} (-1)^{(r-s)(c-(p-r))} \alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \end{aligned}$$

and a trivial mod 2 calculation shows the signs match up.

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator τ introduced above. Let $\mathcal{F}^*$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^* : Tot(\mathcal{F}^* \otimes \mathcal{F}^*) \to \mathcal{F}^*.$$

This means the following: For s a local section of $\mathcal{F}^a$, and t a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab} t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge t$. The composition

$$Tot(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*) \otimes s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)) \longrightarrow s\mathcal{C}^*(\mathcal{U}, Tot(\mathcal{F}^* \otimes_{\mathbf{Z}} \mathcal{F}^*)) \longrightarrow s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$$

induces a cup product on cohomology

$$H^n(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)) \times H^m(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)) \longrightarrow H^{n+m}(s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)).$$

and so in the limit also a product on Cech hypercohomology and therefore (using hypercoverings if needed) a product in hypercohomology of $\mathcal{F}^*$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider an element α of degree n in $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$ and an element β of degree m in $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$ and we compute

$$\begin{aligned}
\tau(\alpha \cup \beta)_{i_0 \ldots i_p} &= (-1)^{p(p+1)/2} (\alpha \cup \beta)_{i_p \ldots i_0} \\
&= (-1)^{p(p+1)/2} \sum\nolimits_{r=0}^{p} (-1)^{(p-r)(m-r)} \alpha_{i_p \ldots i_r} \otimes \beta_{i_r \ldots i_0} \\
&= (-1)^{p(p+1)/2} \sum\nolimits_{r=0}^{p} (-1)^{(p-r)(m-r) + r(r+1)/2 + (p-r)(p-r+1)/2} \tau(\alpha)_{i_r \ldots i_p} \otimes \tau(\beta)_{i_0 \ldots i_r}
\end{aligned}$$

The image of this in $s^{n+m}\mathcal{C}^*(\mathcal{U}, \mathcal{F}^*)$ equals

$$(-1)^{p(p+1)/2} \sum\nolimits_{r=0}^{p} (-1)^{(p-r)(m-r) + r(r+1)/2 + (p-r)(p-r+1)/2 + (m-r)(n-(p-r))} \tau(\beta)_{i_0 \ldots i_r} \wedge \tau(\alpha)_{i_r \ldots i_p}$$

because $\wedge$ is graded commutative. But this is the same as the image of

$$(-1)^{nm} (\tau(\beta) \cup \tau(\alpha))_{i_0 \ldots i_p} = (-1)^{nm} \sum\nolimits_{r=0}^{p} (-1)^{r(n-(p-r))} \tau(\beta)_{i_0 \ldots i_r} \otimes \tau(\alpha)_{i_r \ldots i_p}$$

This proves the desired result since we proved earlier that τ acts as the identity on cohomology.

Suppose that

$$0 \to \mathcal{F}_1^* \to \mathcal{F}_2^* \to \mathcal{F}_3^* \to 0$$

and

$$0 \leftarrow \mathcal{G}_1^* \leftarrow \mathcal{G}_2^* \leftarrow \mathcal{G}_3^* \leftarrow 0$$

are short exact sequences of bounded below complexes of abelian sheaves on X. We will use the following convention and notation: we think of $\mathcal{F}_1^q$ as a subsheaf of $\mathcal{F}_2^q$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if s is a local section of $\mathcal{F}_1^q$ we use s to denote the corresponding section of $\mathcal{F}_2^q$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if s is a local section of $\mathcal{F}_2^q$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^q$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if s is a local section of $\mathcal{F}_2^q$ and $\bar{s} = 0$ then s is a local section of $\mathcal{F}_1^q$. Let $\mathcal{H}^*$ be another complex of abelian sheaves, and suppose we have maps of comlpexes

$$\gamma_i : Tot(\mathcal{F}_i^* \otimes_{\mathbf{Z}} \mathcal{G}_i^*) \longrightarrow \mathcal{H}^*$$

which are compatible with the maps between the complexes. So for example, for local sections s of $\mathcal{F}_2^q$ and t pf $\mathcal{G}_3^{q'}$ we have $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{q+q'}$. In this situation, suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of X. Suppose that α, resp. β is an element of $s^n\mathcal{C}^*(\mathcal{U}, \mathcal{F}_2^*)$, resp. $s^m\mathcal{C}^*(\mathcal{U}, \mathcal{G}_2^*)$ with the property that

$$d(\bar{\alpha}) = 0, \text{ and } d(\bar{\beta}) = 0.$$

This means that

$\alpha_3 = \bar{\alpha}$ is a degree n cocycle in the simple complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}_3^*)$,
$\alpha_1 = d(\alpha)$ is a degree $n+1$ cocycle in the simple complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{F}_1^*)$,
$\beta_1 = \bar{\beta}$ is a degree n cocycle in the simple complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{G}_1^*)$, and
$\beta_3 = d(\beta)$ is a degree $m+1$ cocycle in the simple complex $s\mathcal{C}^*(\mathcal{U}, \mathcal{G}_3^*)$.

I claim that

$$\gamma_1(\alpha_1 \cup \beta_1), \text{ and } \gamma_3(\alpha_3 \cup \beta_3)$$

represent the same cohology class up to sign. The reason is simply that we may compute

$$\begin{aligned} d(\gamma_2(\alpha \cup \beta)) &= \gamma_2(d(\alpha \cup \beta)) \\ &= \gamma_2(d(\alpha) \cup \beta + (-1)^n \alpha \cup d(\beta)) \\ &= \gamma_2(\alpha_1 \cup \beta) + (-1)^n \gamma_2(\alpha \cup \beta_3) \\ &= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3) \end{aligned}$$

So this even tells us that the sign is $(-1)^{n+1}$.