Intersection Theory

This is an old note on intersection theory written for a graduate student seminar in the Fall of 2007 organized by Johan de Jong. In the summer of 2009 a new group of students led by Johan de Jong and Qi You reworked this material which then became a chapter of the Stacks project. We strongly urge the reader to read this online at

https://stacks.math.columbia.edu/tag/0AZ6

instead of reading the old material below. In particular, we do not vouch for the correctness of what follows.

Cycles. Let X be a nonsingular projective variety over an algebraically closed field C. A k-cycle on X is a finite formal sum \( \sum n_i[Z_i] \) where each \( Z_i \) is a closed subvariety of dimension k.

Pushforward. Suppose that \( f : X \to Y \) is a morphism of projective smooth varieties. Let \( Z \subset X \) be a k-dimensional closed subvariety. We define \( f_\ast[Z] \) to be 0 if \( \dim(f(Z)) < k \) and \( d \cdot [f(Z)] \) if \( \dim(f(Z)) = k \) where \( d = [C(Z) : C(f(Z))] \). Let \( \alpha = \sum n_i[Z_i] \) be a k-cycle on Y. The pushforward of \( \alpha \) is the sum \( f_\ast\alpha = \sum n_i f_\ast[Z_i] \) where each \( f_\ast[Z_i] \) is defined as above.

Cycle associated to closed subscheme. Suppose that \( X \) is a nonsingular projective variety and \( Z \subset X \) is a closed subscheme with \( \dim(Z) \leq k \). Let \( Z_i \) be the irreducible components of \( Z \) of dimension k and let \( n_i \) be the length of the local ring of \( Z \) at the generic point of \( Z_i \). We define the k-cycle associated to \( Z \) to be the k-cycle \( [Z]_k = \sum n_i[Z_i] \).

Cycle associated to a coherent sheaf. Suppose that \( X \) is a nonsingular projective variety and \( \mathcal{F} \) is a coherent \( \mathcal{O}_X \)-module on \( X \) with \( \dim(\text{Supp}(\mathcal{F})) \leq k \). Let \( Z_i \) be the irreducible components of \( \text{Supp}(\mathcal{F}) \) of dimension k and let \( n_i \) be the length of the stalk of \( \mathcal{F} \) at the generic point of \( Z_i \). We define the k-cycle associated to \( \mathcal{F} \) to be the k-cycle \( [\mathcal{F}]_k = \sum n_i[Z_i] \).

Note that, if \( \dim(Z) \leq k \), then \( [Z]_k = [\mathcal{O}_Z]_k \).

Suppose that \( f : X \to Y \) is a morphism of projective smooth varieties. Let \( Z \subset X \) be a k-dimensional closed subvariety. It can be shown that \( f_\ast[Z] = [f_\ast\mathcal{O}_Z]_k \). See [Serre, Chapter V].

Flat pullback. Suppose that \( f : X \to Y \) is a flat morphism of nonsingular projective varieties of relative dimension r, in other words all fibres have dimension r. Let \( Z \subset X \) be a k-dimensional closed subvariety. We define \( f^\ast[Z] \) to be the \( k+r \)-cycle associated to the scheme theoretic inverse image: \( f^\ast[Z] = [f^\ast(\mathcal{O}_Z)]_{k+r} \).

Let \( \alpha = \sum n_i[Z_i] \) be a k-cycle on Y. The pullback of \( \alpha \) is the sum \( f^\ast\alpha = \sum n_i f^\ast[Z_i] \) where each \( f^\ast[Z_i] \) is defined as above.

With this notation, we get that \( f^\ast[\mathcal{F}]_k = [f^\ast\mathcal{F}]_{k+r} \) if \( \mathcal{F} \) is a coherent sheaf on \( Y \) and the dimension of the support of \( \mathcal{F} \) is at most k.

Intersection multiplicities using Tor formula. Suppose that \( X \) is a nonsingular projective variety and that \( W, V \subset X \) are closed subvarieties with \( \dim(W) = s \) and \( \dim(V) = r \). Assume that \( \dim(W \cap V) \leq r+s-\dim(X) \). We say that \( W \) and \( V \) intersect properly if this holds. In this case the sheaves \( Tor_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V) \) are coherent, supported on \( V \cap W \), and zero if \( j < 0 \) or \( j > \dim(X) \). We define

\[
W \cdot V = \sum_i (-1)^i [Tor_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)]_{r+s-\dim(X)}.
\]

With this notation, the cycle \( V \cdot W \) is a formal linear combination \( \sum e_i Z_i \) of the irreducible components \( Z_i \) of the intersection \( V \cap W \). The integers \( e_i \) are called the intersection multiplicities: \( e_i = e(X, V \cdot W, Z_i) \). They satisfy many good properties, see [Serre].

Computing intersection multiplicities. In the situation above, let \( Z = Z_i \) be one of the irreducible components. Let \( A \) be the local ring of \( X \) at the generic point of \( Z \). Suppose that the ideal of \( V \) in \( A \) is cut out by a regular sequence \( x_1, \ldots, x_c \) and suppose that the local ring of \( W \) at the generic point of \( Z \) corresponds to the quotient map \( A \to B \). In this case \( e(X, V \cdot W, Z) \) is equal to \( c! \) times the leading coefficient in the Hilbert polynomial

\[
t \mapsto \text{length}_A(B/(x_1, \ldots, x_c)^t B), \quad t \gg 0.
\]
Consider the case that $c = 1$, i.e., $V$ is a(n effective) Cartier divisor. Then $x_1$ is a nonzero divisor on $B$ by properness of intersection of $V$ and $W$. We easily deduce
\[
e(X, V \cdot W, Z) = \text{length}_A(B/x_1 B).
\]

More generally, if the local ring $B$ is Cohen-Macaulay, then we have
\[
e(X, V \cdot W, Z) = \text{length}_A(B/x_1 B + \ldots + x_C B).
\]

**Intersection product using Tor formula.** Suppose that $X$ is a nonsingular projective variety. Suppose $\alpha = \sum n_i [W_i]$ is an $r$-cycle, and $\beta = \sum j m_j [V_j]$ is an $s$-cycle on $X$. We say that $\alpha$ and $\beta$ **intersect properly** if $W_i$ and $V_j$ intersect properly for all $i$ and $j$. In this case we define
\[
\alpha \cdot \beta = \sum_{i,j} n_i m_j W_i \cdot V_j,
\]
where $W_i \cdot V_j$ is as defined above using the Tor-formula.

Suppose $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on $X$ with $\dim(\text{Supp}(\mathcal{F})) \leq s$, $\dim(\text{Supp}(\mathcal{G})) \leq r$ and $\dim(\text{Supp}(\mathcal{F}) \cap \text{Supp}(\mathcal{G})) \leq r + s - \dim X$. In this case
\[
[\mathcal{F}]_s \cdot [\mathcal{G}]_r = \sum (-1)^i [\text{Tor}_i^O(\mathcal{F}, \mathcal{G})]_{r+s-\dim X}.
\]

See [Serre, Chapter V].

**Exterior product.** Let $X$ and $Y$ be nonsingular projective varieties. Let $\nu$, resp. $W$ be a closed subvariety of $X$, resp. $Y$. The product $V \times W$ is a closed subscheme of $X \times Y$. It is a subvariety because the ground field is algebraically closed. For a $k$-cycle $\alpha = \sum n_i [V_i]$ and a $l$-cycle $\beta = \sum m_j [V_j]$ on $Y$ we define $\alpha \times \beta = \sum n_i m_j [V_i \times W_j]$.

Consider the subvariety $X \subset X$ with class $[X]$. Note that $pr_X^*(\beta) = [X] \times \beta$. Note that $\alpha \times [Y]$ and $[X] \times \beta$ intersect properly on $X \times Y$. With the definitions above we have $\alpha \times \beta = (\alpha \times [Y]) \cdot ([X] \times \beta) = pr_X^*(\alpha) \cdot pr_Y^*(\beta)$.

**Reduction to the diagonal.** Let $X$ be a nonsingular projective variety. Let $\Delta \subset X \times X$ denote the diagonal. We will identify $\Delta$ with $X$. Let $\alpha$, resp. $\beta$ be $r$-cycles, resp. $s$-cycles on $X$. Assume $\alpha$ and $\beta$ intersect properly. In this case $\alpha \times \beta$ and $[\Delta]$ intersect properly. Note that the cycle $\Delta \cdot \alpha \times \beta$ is supported on the diagonal and hence we can think of it as a cycle on $X$. With this convention we have $\alpha \cdot \beta = \Delta \cdot \alpha \times \beta$.

See [Serre, Chapter V].

Perhaps a less confusing formulation would be that $pr_1,*(\Delta \cdot \alpha \times \beta) = \alpha \cdot \beta$, where $pr_1 : X \times X \to X$ is the projection.

**Flat pullback and intersection products.** Suppose that $f : X \to Y$ is a flat morphism of nonsingular projective varieties. Suppose that $\alpha$ is a $k$-cycle on $Y$ and that $\beta$ is a $l$-cycle on $Y$. Assume that $\alpha$ and $\beta$ intersect properly. Then $f^* \alpha$ and $f^* \beta$ intersect properly and $f^*(\alpha \cdot \beta) = f^* \alpha \cdot f^* \beta$. This is not hard to see from the material above.

**Projection formula for flat maps.** Let $f : X \to Y$ be a flat morphism of relative dimension $r$ of nonsingular projective varieties. Let $\alpha$ be an $k$-cycle on $X$ and let $\beta$ be a $l$-cycle on $Y$. Assume that $\alpha$ and $\beta$ intersect properly. Then $f_* \alpha$ and $f_* \beta$ intersect properly and $f_*(\alpha \cdot \beta) = f_* \alpha \cdot f_* \beta$. This is not hard to see from the material above.

We explain how to prove the projection formula in the flat case. Let $W \subset X$ be a closed subvariety of dimension $k$. Let $V \subset Y$ be a closed subvariety of dimension $l$, so $f^{-1}(V)$ has pure dimension $l + r$. Assume that $W$ and $[f^{-1}(V)]$ intersect properly. Note that $f(W \cap f^{-1}(V)) = f(W) \cap V$. Hence it follows that $f(W)$ and $V$ intersect properly as well. Let $Z \subset f(W) \cap V$ be an irreducible component of dimension $k + l - \dim Y$. Let $Z_i \subset W \cap f^{-1}(V)$ be the irreducible components of $W \cap f^{-1}(V)$ dominating $Z$. Let $A$ be the local ring of $X$ at the generic point of $Z$. Let $A_i$ be the local ring of $Y$ at the generic point of $Z_i$. Let $B$ be the local ring...
ring of \( f(W) \) at the generic point of \( Z \). Let \( B' \) be the stalk of \( f_*({\mathcal O}_W) \) at the generic point of \( Z \). Then \( B \to B' \) is finite, \( B' \) is semi-local, and the localizations \( B'_i \) of \( B' \) are the local rings of \( W \) at the generic point of the \( Z_i \). Thus they are quotients \( A_i \to B'_i \). Let \( C \) be the local ring of \( V \) at the generic point of \( Z \). The multiplicity of \( Z \) in \( f_*([W]) : V \) is by definition
\[
(I) = [B' : B] \sum (-1)^i \text{length}_{A_i}(\text{Tor}^{A_i}_j(B', C)).
\]
Here \([B' : B]\) is the rank of the \( B \)-module \( B' \). The multiplicity of \( Z \) in \( f_*([W] \cdot f^*[V]) \) is by definition
\[
(II) = \sum_k (-1)^i \text{length}_{A_i}(\text{Tor}^{A_i}_j(B'_1, A_i \otimes_A C)) [\kappa(A_i) : \kappa(A)]
\]
Here \( \kappa(-) \) indicates the residue field. The first thing is to note that \( \text{length}_{A_i}(M) = [\kappa(A_i) : \kappa(A)] \text{length}_{A_i}(M) \) for a finite length \( A_i \)-module \( M \). We can compute all the Tor groups by choosing a free resolution of \( C \) as an \( A \)-module. Doing this it is easy to see that \((I) \) equals \( \sum (-1)^i \text{length}_{A}(\text{Tor}^{A}_j(B', C)) \). Finally, note that, by definition, there is an \( A \)-module map \( B^\oplus \to B' \) whose kernel and cokernel are supported in a proper closed subset of \( \text{Spec}(B) \). From the additivity properties of the Tor-formula, see [Serre, Chapter V], it follows that \( \sum (-1)^i \text{length}_{A}(\text{Tor}^{A}_j(B', C)) = [B' : B] \sum (-1)^i \text{length}_{A}(\text{Tor}^{A}_j(B, C)) \) as desired.

**Rational Equivalence.** Let \( X \) be a nonsingular projective variety. Let \( \alpha = \sum_i n_i [W_i] \) be a \((k+1)\)-cycle on \( X \times \mathbf{P}^1 \), and let \( a, b \) be two closed points of \( \mathbf{P}^1 \). Assume that \( X \times a \) and \( \alpha \) intersect properly, and that \( X \times b \) and \( \alpha \) intersect properly. This will be the case if each \( W_i \) dominates \( \mathbf{P}^1 \) for example. Let \( pr_X : X \times \mathbf{P}^1 \to X \) be the projection morphism. A cycle rationally equivalent to zero is any cycle of the form
\[
pr_X (\alpha \times X \times a - \alpha \times X \times b).
\]
This is a \( k \)-cycle. Note that these cycles are easy to compute in practice (given \( \alpha \)) because they are obtained by proper intersection with Cartier divisors (see formula above). It is a fact that the collection of \( k \)-cycles rationally equivalent to zero is a additive subgroup of the group of \( k \)-cycles. We say two \( k \) cycles are *rationally equivalent*, notation \( \alpha \sim_{\text{rat}} \alpha' \) if \( \alpha - \alpha' \) is a cycle rationally equivalent to zero. See Chapter I of [Fulton].

**Pushforward and rational equivalence.** Suppose that \( f : X \to Y \) is a morphism of projective smooth varieties. Let \( \alpha \sim_{\text{rat}} 0 \) be a \( k \)-cycle on \( X \) rationally equivalent to 0. Then the pushforward of \( \alpha \) is rationally equivalent to zero: \( f_* \alpha \sim_{\text{rat}} 0 \). See Chapter I of [Fulton].

**Pullback and rational equivalence.** Suppose that \( f : X \to Y \) is a flat morphism of relative dimension \( r \) of projective smooth varieties. Let \( \alpha \sim_{\text{rat}} 0 \) be a \( k \)-cycle on \( Y \) rationally equivalent to 0. Then the pullback of \( \alpha \) is rationally equivalent to zero: \( f^* \alpha \sim_{\text{rat}} 0 \). See Chapter I of [Fulton].

**Moving Lemma.** The moving lemma states that given an \( r \)-cycle \( \alpha \) and a \( s \) cycle \( \beta \) there exists \( \alpha', \alpha' \sim_{\text{rat}} \alpha \) such that \( \alpha \) and \( \beta \) intersect properly. See [Samuel], [Chevalley], or [Fulton, Example 11.4,1].

**Intersection product and rational equivalence.** With definitions as above we show that the intersection product is well defined modulo rational equivalence. Let \( X \) be a nonsingular projective algebraic variety. Let \( \alpha \), resp. \( \beta \) be a \( s \), resp. \( r \) cycle on \( X \). Assume that \( \alpha \) and \( \beta \) intersect properly so that \( \alpha \cdot \beta \) is defined. Finally, assume that \( \alpha \sim_{\text{rat}} 0 \). Goal: show that \( \alpha \cdot \beta \sim_{\text{rat}} 0 \).

After some formal arguments this amounts to showing the following statement. Let \( W \subset X \times \mathbf{P}^1 \) be a \((s+1)\)-dimensional subvariety dominating \( \mathbf{P}^1 \). Let \( W_a \), resp. \( W_b \) be the fibre of \( W \to \mathbf{P}^1 \) over \( a \), resp. \( b \). Let \( V \) be a \( r \)-dimensional subvariety of \( X \) such that \( V \) intersects both \( W_a \) and \( W_b \) properly. Then \( V \cdot [W_a] \sim_{\text{rat}} V \cdot [W_b] \).
In order to see this, note first that \([W_a] = pr_{X,*}(W \times X \times a) \) and similar for \([W_b] \). Thus we reduce to showing
\[
V \cdot pr_{X,*}(W \times X \times a) \sim_{\text{rat}} V \cdot pr_{X,*}(W \times X \times b).
\]
The projection formula – which may be applied – says \( V \cdot pr_{X,*}(W \times X \times a) = pr_{X,*}(V \times \mathbf{P}^1 \cdot (W \times X \times a)) \), and similar for \( b \). Thus we reduce to showing
\[
pr_{X,*}(V \times \mathbf{P}^1 \cdot (W \times X \times a)) \sim_{\text{rat}} pr_{X,*}(V \times \mathbf{P}^1 \cdot (W \times X \times b))
\]
Associativity for the intersection multiplicities (see [Serre, Chapter V]) implies that $V \cdot (W \cdot X \times a) = (V \times \mathbb{P}^1 \cdot W) \cdot X \times a$ and similar for $b$. Thus we reduce to showing

$$pr_{X,*}((V \times \mathbb{P}^1 \cdot W) \cdot X \times a) \sim_{rat} pr_{X,*}((V \times \mathbb{P}^1 \cdot W) \cdot X \times b)$$

which is true by definition of rational equivalence.

**Upshot: Chow rings.** Using the above, for any nonsingular projective $X$ we set $A_k(X)$ equal to the group of $k$-cycles on $X$ modulo rational equivalence. Since it is more convenient we also use $A^c(X) = A_{\dim X - c}(X)$ to denote the group of codimension $c$ cycles modulo rational equivalence. The intersection product defines a product

$$A^k(X) \times A^l(X) \to A^{k+l}(X)$$

defined as follows: for $a \in A^k(X)$ and $b \in A^l(X)$ we can find a codimension $k$ cycle $\alpha$ representing $a$, a codimension $l$ cycle $\beta$ representing $b$ such that $\alpha$ and $\beta$ intersect properly. We define $a \cdot b$ to be the rational equivalence class of $\alpha \cdot \beta$. End result: A commutative and associative graded ring $A^*(X)$ with unit $1 = [X]$.

**Pullback for a general morphism.** Let $X$ and $Y$ be nonsingular projective varieties, and let $f : X \to Y$ be a morphism. We define

$$f^* : A_k(Y) \to A_{k+\dim X - \dim Y}(X)$$

by the rule

$$f^*(\alpha) = pr_{X,*}(\Gamma_f \cdot pr_Y^*(\alpha))$$

where $\Gamma_f \subset X \times Y$ is the graph of $f$. Note that it is defined only on cycle classes and not on cycles. This pullback satisfies:

1. $f^* : A^*(Y) \to A^*(X)$ is a ring map,
2. $(f \circ g)^* = g^* \circ f^*$ for a composable pair $f, g$,
3. the projection formula holds: $f_* (\alpha) \cdot \beta = f_* (\alpha \cdot f^* \beta)$, and
4. if $f$ is flat then it agrees with the previous definition.

All of these follow easily from the above. For (1) you have to show that $pr_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) = pr_{X,*}(\Gamma_f \cdot \alpha) \cdot pr_{X,*}(\Gamma_f \cdot \beta)$. It is easy to see that if $\alpha$ intersects $\Gamma_f$ properly, then $\Gamma_f \cdot \alpha = \Gamma_f \cdot pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha))$ as cycles because $\Gamma_f$ is a graph. Thus we get

$$pr_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) = pr_{X,*}(\Gamma_f \cdot pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha)) \cdot \beta)$$

$$= pr_{X,*}(pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha)) \cdot (\Gamma_f \cdot \beta))$$

$$= pr_{X,*}(\Gamma_f \cdot \alpha) \cdot pr_{X,*}(\Gamma_f \cdot \beta)$$

the last step by the projection formula in the flat case. Properties (2) and (3) are formal [for (3) use the flat projection formula twice]. Property (4) rests on identifying the intersection product $\Gamma_f \cdot \alpha$ in the case $f$ is flat.

**Pullback of cycles.** Suppose that $X$ and $Y$ be nonsingular projective varieties, and let $f : X \to Y$ be a morphism. Suppose that $Z \subset Y$ is a closed subvariety. Let $f^{-1}(Z)$ be the scheme theoretic inverse image:

$$f^{-1}(Z) \to Z$$

$$\downarrow \quad \downarrow$$

$$X \to Y$$

is a fibre product diagram of schemes. In particular $f^{-1}(Z) \subset X$ is a closed subscheme of $X$. In this case we always have

$$\dim f^{-1}(Z) \geq \dim Z + \dim X - \dim Y.$$  

If equality holds in the formula above, then $f^*[Z] = [f^{-1}(Z)]_{\dim Z + \dim X - \dim Y}$ provided that the scheme $Z$ is Cohen-Macaulay at the images of the generic points of $f^{-1}(Z)$. This follows by identifying $f^{-1}(Z)$ with the scheme theoretic intersection of $\Gamma_f$ and $X \times Z$ and using the computation ($*$) of the intersection multiplicities we gave above.
References

[Chevalley] Less classes d’équivalence rationelle, I, II
[Fulton] Intersection Theory